Deformation theory of singular symplectic n-folds

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Introduction

By a symplectic manifold (or a symplectic n-fold) we mean a compact Kaehler manifold of even dimension n with a non-degenerate holomorphic 2-form ω, i.e. $\omega^{n/2}$ is a nowhere-vanishing n-form. This notion is generalized to a variety with singularities. We call X a projective symplectic variety if X is a normal projective variety with rational Gorenstein singularities and if the regular locus U of X admits a non-degenerate holomorphic 2-form ω. A symplectic variety will play an important role together with a singular Calabi-Yau variety in the generalized Bogomolov decomposition conjecture. Now that essentially a few examples of symplectic manifolds are discovered, it seems an important task to seek new symplectic manifolds by deforming symplectic varieties. In this paper we shall study a projective symplectic variety from a view point of deformation theory. If X has a resolution $\pi : \tilde{X} \to X$ such that $(\tilde{X}, \pi^*\omega)$ is a symplectic manifold, we say that X has a symplectic resolution. Our first results are concerned with a birational contraction map of a symplectic manifold.

Proposition (1.4). *Let $\pi : \tilde{X} \to X$ be a birational projective morphism from a projective symplectic n-fold $\tilde{X}$ to a normal n-fold X. Let S_i be the set of points $p \in X$ such that $\dim \pi^{-1}(p) = i$. Then $\dim S_i \leq n - 2i$. In particular, $\dim \pi^{-1}(p) \leq n/2$.*

Proposition (1.6). *Let $\pi : \tilde{X} \to X$ be a birational projective morphism from a projective symplectic n-fold $\tilde{X}$ to a normal n-fold X. Then X has only canonical singularities and its dissident locus Σ_0 has codimension at least 4 in X. Moreover, if $\Sigma \setminus \Sigma_0$ is non-empty, then $\Sigma \setminus \Sigma_0$ is a disjoint union of smooth varieties of dim $n-2$ with everywhere non-degenerate 2-forms.*

When X has only an isolated singularity $p \in X$, every irreducible component of $\pi^{-1}(p)$ is Lagrangian (Proposition (1.11)). In this situation it is conjectured that the exceptional locus is isomorphic to $\mathbf{P}^{n/2}$ with normal bundle $\Omega^1_{\mathbf{P}^{n/2}}$. Similar results to (1.4) and (1.11) are obtained independently by Wierzba [Wi].

We shall exhibit four examples of birational contraction maps of symplectic 4-folds in (1.7). The second example shows that the Kaehler (projective) assumption of a symplectic manifold is not necessarily preserved under an elementary transformation. The fourth example deals with a symplectic manifold

obtained as a resolution of certain quotient of a Fano scheme of lines on a cubic 4-fold. As for a fiber space structure of a symplectic n-fold, see [Ma].

After we study the birational contraction map of a symplectic manifold in section 1, we shall prove our main theorem in section 2:

Theorem (2.2). *Let $\pi : \tilde{X} \to X$ be a symplectic resolution of a projective symplectic variety X of dimension n. Then the Kuranishi spaces* $\mathrm{Def}(\tilde{X})$ *and* $\mathrm{Def}(X)$ *are both smooth of the same dimension. There exists a natural map* $\pi_* : \mathrm{Def}(\tilde{X}) \to \mathrm{Def}(X)$ *and π_* is a finite covering* [1]*. Moreover, X has a flat deformation to a smooth symplectic n-fold X_t. Any smoothing X_t of X is a symplectic n-fold obtained as a flat deformation of $\tilde{X}$.*

(2.2) was proved by Burns-Wahl [B-W] for K3 surfaces. Given a one-parameter flat deformation $f : \mathcal{X} \to \Delta$ of such X as (2.2), by Theorem, we could have a simultaneous resolution $\nu : \tilde{\mathcal{X}} \to \mathcal{X}'$ after a suitable finite base change $\mathcal{X}' \to \Delta'$ of $\mathcal{X}$ by $\Delta' \to \Delta$.

The same situation as (2.2) naturally arises for Calabi-Yau 3-folds; but the results for them are very partial as compared with symplectic case (cf. Example (2.4)).

On the other hand, it is natural to consider a symplectic variety which does not have a symplectic resolution; for example, such varieties appear in a work of O'Grady [O] as the moduli spaces of rank 2 semi-stable sheaves on a K3 surface with $c_1 = 0$ and with even $c_2 \geq 6$. At the moment it is not clear when these varieties have flat deformations to symplectic manifolds. But we can prove that such varieties have unobstructed deformations:

Theorem (2.5). *Let X be a projective symplectic variety. Let $\Sigma \subset X$ be the singular locus. Assume that* $\mathrm{codim}(\Sigma \subset \mathrm{X}) \geq 4$*. Then* $\mathrm{Def}(X)$ *is smooth.*

We shall give a rough sketch of the proof of Theorem (2.2) in the remainder.

First note that X has only rational Gorenstein singularities. Then the existence of the map π_* follows from the fact that $R^1\pi_*\mathcal{O}_{\tilde{X}} = 0$ (cf. [Ko-Mo, (11.4)]).

Let U be the complement of Σ_0 in X and write $\tilde{U}$ for $\pi^{-1}(U)$. By (1.4) and (1.6), we can prove, roughly speaking, that a deformation of $\tilde{X}$ (resp. X) is equivalent to that of $\tilde{U}$ (resp. U). (See Proposition (2.1).) From this fact it follows that $\pi_* : \mathrm{Def}(\tilde{X}) \to \mathrm{Def}(X)$ is finite.

Finally we compare the dimensions of tangent spaces of $\mathrm{Def}(X)$ and $\mathrm{Def}(\tilde{X})$ at the origin and then conclude that $\mathrm{Def}(X)$ is smooth. Since $\mathrm{Def}(\tilde{X})$ is smooth by Bogomolov [Bo], we only have to prove that $\dim \mathbf{T}^1_X = \dim \mathbf{T}^1_U$ is not larger than $h^1(\tilde{X}, \Theta_{\tilde{X}}) = h^1(\tilde{U}, \Theta_{\tilde{U}})$. We need here a detailed description of the sheaf $T^1_U := \underline{Ext}^1(\Omega^1{}_U, \mathcal{O}_U)$ (Lemma (1.9), Corollary (1.10)).

The last statement will be proved in the following way. By the existence of a non-degenerate 2-form ω, there is an obstruction to extending a holomorphic

[1] Precisely, there are open subsets $0 \in V \subset \mathrm{Def}(\tilde{X})$ and $0 \in W \subset \mathrm{Def}(X)$ such that $\pi_*|_V : V \to W$ is a proper surjective map with finite fibers.

curve on $\tilde{X}$ sideways in a given one-parameter small deformation $\tilde{\mathcal{X}} \to \Delta^1$. Therefore, if we take a general curve of $\mathrm{Def}(\tilde{X})$ passing through the origin and take a corresponding small deformation of $\tilde{X}$, then no holomorphic curves survive (cf. [Fu, Theorem (4.8)]).

Let $t \in \mathrm{Def}(X)$ be a generic point (that is, t is outside the union of a countable number of proper subvarieties of $\mathrm{Def}(X)$). Since $\pi_* : \mathrm{Def}(\tilde{X}) \to \mathrm{Def}(X)$ is a finite covering, we may assume that X_t has a symplectic resolution $\pi_t : \tilde{X}_t \to X_t$. By the argument above, $\tilde{X}_t$ contains no curves. By Chow lemma [Hi], there is a bimeromorphic projective map $h : W \to X_t$ such that h is factored through π_t. Since $h^{-1}(p)$ is the union of projective varieties for any point $p \in X_t$, ${\pi_t}^{-1}(p)$ is the union of Moishezon varieties. If π_t is not an isomorphism, ${\pi_t}^{-1}(p)$ has positive dimension for some $p \in X_t$; hence $\tilde{X}_t$ contains curves, which is a contradiction. Thus π_t is an isomorphism and X_t is a (smooth) symplectic n-fold.

The author thanks A. Fujiki for giving him invaluable informations on this topic. The first version of this paper was written in 1998. After that the author was informed that Wierzba [Wi] independently obtained similar results to (1.4) and (1.11).

1. Birational contraction maps of symplectic n-folds

A symplectic n-fold means a symplectic *manifold* of dimension n. We shall state three lemmas which will be used later. The first lemma is essentially a linear algebra.

Lemma (1.1). *Let V be a complex manifold with $\dim V = 2r$ and let ω be an everywhere non-degenerate holomorphic 2-form on V (i.e. $\wedge^r\omega$ is nowhere-vanishing.) Let E be a subvariety of V with $\dim E > r$. Then $\omega|_E$ is a non-zero 2-form on E.*

Lemma (1.2). *Let $f : V \to W$ be a birational projective morphism from a complex manifold V to a normal variety W. Let $p \in W$ and assume that the germ (W, p) of W at p has rational singularities. Assume that $E := f^{-1}(p)$ is a simple normal crossing divisor of V. Then $H^0(E, \hat{\Omega}^i_E) = 0$ for $i > 0$, where $\hat{\Omega}^i_E := \Omega^i_E/(torsion)$.*

Proof. Denote by $F^{\cdot}$ (resp. $W_{\cdot}$) the Hodge filtration (resp. weight filtration) of $H^i(E) := H^i(E, \mathbf{C})$. Note that these two filtrations give a mixed Hodge structure on $H^i(E)$. Since E is a proper algebraic scheme, $Gr^W_j(H^i(E)) = 0$ for $j > i$.

Assume that $H^0(E, \hat{\Omega}^i_E) \neq 0$. Then $Gr^i_F Gr^W_i(H^i(E)) \neq 0$. By the Hodge symmetry $Gr^0_F Gr^W_i(H^i(E)) \neq 0$, and hence $Gr^0_F(H^i(E)) = H^i(E, \mathcal{O}_E) \neq 0$.

On the other hand, $(R^i f_* \mathcal{O}_V)_p = 0$ for $i > 0$ because (W, p) has only rational singularities. Take a sufficiently small open neighborhood V' of $f^{-1}(p)$ in V. There exists a commutative diagram of Hodge spectral sequences

$$
\begin{array}{ccc}
H^k(V', \Omega^j_{V'}) & \Longrightarrow & H^{j+k}(V', \mathbf{C}) \\
\downarrow & & \downarrow \\
H^k(E, \hat{\Omega}^j_E) & \Longrightarrow & H^{j+k}(E, \mathbf{C})
\end{array}
\tag{1}
$$

Note that $H^i(V', \mathbf{C}) \cong H^2(E, \mathbf{C})$. Denote by $F_1^{\cdot}$ (resp. $F_2^{\cdot}$) the filtrations on $H^i(V', \mathbf{C})$ (resp. $H^i(E, \mathbf{C})$) induced by the spectral sequences. There is a natural surjection $Gr^0_{F_1} H^i(V', \mathbf{C}) \to Gr^0_{F_2} H^i(E, \mathbf{C})$. As $(R^i f_* \mathcal{O}_V)_0 = 0$ for $i > 0$, $H^i(V', \mathcal{O}_{V'}) = 0$. Therefore we have $Gr^0_{F_1} H^2(V', \mathbf{C}) = Gr^0_{F_2} H^i(E, \mathbf{C}) = 0$. Since the second spectral sequence degenerates at E_1 terms, $H^i(E, \mathcal{O}_E) = 0$, which is a contradiction.

Lemma(1.3). *Let V be a symplectic n-fold and let H be a smooth 3-dimensional subvariety of V containing a smooth rational curve C with $N_{H/V}|_C \cong \mathcal{O}^{\oplus n-3}$. Assume that $N_{C/H} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, $\mathcal{O}(-2) \oplus \mathcal{O}$ or $\mathcal{O}(-3) \oplus \mathcal{O}(1)$. Then* $\mathrm{Hilb}(V)$ *is smooth of dimension (n-2) at [C]. Moreover, in this case, $N_{C/V} \cong \mathcal{O}^{\oplus(n-2)} \oplus \mathcal{O}(-2)$ or $\mathcal{O}^{\oplus(n-4)} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-2)$.*

Proof. We shall prove that the Hilbert scheme (functor) has the T^1-lifting property at [C]; then $\mathrm{Hilb}(V)$ is smooth at [C] by [Ra, Ka 1]. Let S_m be the spectrum of the Artinian ring $A_m = \mathbf{C}[t]/(t^{m+1})$. Set $V_m := V \times_{S_0} S_m$. Let $C_m \subset V_m$ be an infinitesimal displacement of C to m-th order, and let $C_{m-1} := C_m \times_{V_m} V_{m-1}$. We have to prove that

$$H^0(N_{C_m/V_m}) \to H^0(N_{C_{m-1}/V_{m-1}})$$

is surjective. Let ω be a non-degenerate 2-form on V. Then ω lifts to an element $\omega_m \in H^0(\Omega^2_{V_m/S_m})$ in such a way that $\wedge^{n/2}\omega_m \in H^0(\Omega^n_{V_m/S_m})$ is a nowhere vanishing section (that is, if we identify $H^0(\Omega^n_{V_m/S_m})$ with A_m, then $\wedge^{n/2}\omega_m$ corresponds to an invertible emement of A_m). The 2-form ω_m induces a pairing

$$\Theta_{V_m/S_m}|_{C_m} \times \Theta_{V_m/S_m}|_{C_m} \to \mathcal{O}_{C_m}.$$

Since this pairing vanishes on $\Theta_{C_m/S_m} \times \Theta_{C_m/S_m}$ and since ω_m is non-degenerate, one has a surjection

$$\alpha_m : N_{C_m/V_m} \to \Omega^1_{C_m/S_m}$$

by the exact sequence

$$0 \to \Theta_{C_m/S_m} \to \Theta_{V_m/S_m}|_{C_m} \to N_{C_m/V_m} \to 0.$$

Let us first consider the case when $m = 0$. By assumption, we have $N_{H/V}|_C \cong \mathcal{O}^{\oplus n-3}$; hence by the exact sequence

$$0 \to N_{C/H} \to N_{C/V} \to N_{H/V}|_C \to 0$$

we see that $N_{C/V}$ is isomorphic to $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\oplus\mathcal{O}^{\oplus(n-3)}$, $\mathcal{O}(-3)\oplus\mathcal{O}(1)\oplus\mathcal{O}^{\oplus(n-3)}$, $\mathcal{O}(-2)\oplus\mathcal{O}(-1)\oplus\mathcal{O}(1)\oplus\mathcal{O}^{\oplus(n-4)}$ or $\mathcal{O}(-2)\oplus\mathcal{O}^{\oplus(n-2)}$. By the existence of the surjection α_0, the first two cases are excluded. In particular, it is checked that $H^1(\mathrm{Ker}(\alpha_0)) = 0$. Note that this implies that $H^1(\mathrm{Ker}(\alpha_{\mathrm{m}})) = 0$ for all m because there are exact sequences $0 \to \mathrm{Ker}(\alpha_0) \to \mathrm{Ker}(\alpha_{\mathrm{m}}) \to \mathrm{Ker}(\alpha_{\mathrm{m}-1}) \to 0$.

Next consider the following commutative diagram with exact columns and exact rows

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^0(\mathrm{Ker}(\alpha_0)) & \longrightarrow & H^0(\mathrm{Ker}(\alpha_{\mathrm{m}})) & \xrightarrow{\psi_m} & H^0(\mathrm{Ker}(\alpha_{\mathrm{m}-1})) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^0(N_{C/V}) & \longrightarrow & H^0(N_{C_m/V_m}) & \xrightarrow{\phi_m} & H^0(N_{C_{m-1}/V_{m-1}}) \\
 & & \tau_0\downarrow & & \tau_m\downarrow & & \tau_{m-1}\downarrow \\
0 & \longrightarrow & H^0(\Omega^1_C) & \longrightarrow & H^0(\Omega^1_{C_m/S_m}) & \xrightarrow{\varphi_m} & H^0(\Omega^1_{C_{m-1}/S_{m-1}})
\end{array} \quad (2)$$

As we remarked above, $H^1(\mathrm{Ker}(\alpha_{\mathrm{m}})) = 0$ for all m, hence τ_0, τ_m and τ_{m-1} are surjective. By the same reason, ψ_m is also surjective. By the Hodge theory, φ_m is surjective. It now follows from the diagram above that ϕ_m is surjective. Thus the Hilbert scheme $\mathrm{Hilb}(V)$ is smooth at [C]. Its dimension equals $h^0(N_{C/V}) = n-2$.

Proposition (1.4). *Let $\pi : \tilde{X} \to X$ be a birational projective morphism from a projective symplectic n-fold $\tilde{X}$ to a normal n-fold X. Let S_i be the set of points $p \in X$ such that $\dim \pi^{-1}(p) = i$. Then $\dim S_i \leq n-2i$. In particular, $\dim \pi^{-1}(p) \leq n/2$.*

Proof. For a non-empty S_i, we take an irreducible component R_i of $\pi^{-1}(S_i)$ in such a way that

(1) by π, R_i dominates an irreducible component of S_i with $\dim S_i$, and

(2) a general fiber of $R_i \to \pi(R_i)$ has dimension i.

Put $l := n - \dim R_i$. By definition $\dim S_i = n-l-i$. We shall prove that $\dim S_i \geq n-2l$; if this holds, then $i \leq l$, and hence $\dim S_i \leq n-2i$. When $l \geq n/2$, then clearly $\dim S_i \geq n-2l$. We assume that $l < n/2$. We shall derive a contradiction assuming that $\dim S_i < n-2l$ and assuming that S_i is irreducible. When S_i is not irreducible, it is enough only to replace S_i by $\pi(R_i)$.

Take a birational projective morphism $\nu : Y \to \tilde{X}$ in such a way that $F := (\pi\circ\nu)^{-1}(S_i)$ becomes a divisor of a smooth n-fold Y with normal crossings. Set $f = \pi\circ\nu$.

A non-degenerate 2-form ω on $\tilde{X}$ is restricted to a non-zero 2-form on R_i because $\dim R_i > n/2$ (Lemma (1.1)). Therefore we have a non-zero element $\nu^*\omega|_F \in H^0(F, \hat{\Omega}^2_F)$.

For a general point $p \in S_i$, the fiber F_p of the map $F \to S_i$ is a normal crossing variety. Hence, if we take a suitable open set U_i of S_i and replace F by $(\pi \circ \nu)^{-1}(U_i)$, then the sheaf $\hat{\Omega}^2_F$ has a filtration $f^*\Omega^2_{S_i} \subset \mathcal{F} \subset \hat{\Omega}^2_F$ with the exact sequences

$$0 \to \mathcal{F} \to \hat{\Omega}^2_F \to \hat{\Omega}^2_{F/S_i} \to 0$$

$$0 \to f^*\Omega^2_{S_i} \to \mathcal{F} \to f^*\Omega^1_{S_i} \otimes \hat{\Omega}^1_{F/S_i} \to 0$$

Let us prove that the 2-form $\nu^*\omega|_F$ is not the pull-back of any 2-form on S_i. Write $n = 2r$. Assume that $\omega' := \nu^*\omega|_F$ is the pull-back of a 2-form on S_i. Then $\wedge^{r-l}\omega' = 0$ because $\dim S_i < 2r - 2l$. On the other hand, take a general point $q \in R_i$. Since R_i is a submanifold of $\tilde{X}$ of codimension l around q, it is checked by linear algebra that $\wedge^{r-l}(\omega|_{R_i}) \neq 0$ in a open neighborhood of $q \in R_i$. Let F' be the union of irreducible components of F which dominate R_i by ν. Since $\nu^*\omega|_{F'} = (\nu|_{F'})^*(\omega|_{R_i})$, $\wedge^{r-l}(\nu^*\omega|_{F'}) \neq 0$. Since $(\wedge^{r-l}\omega')|_{F'} = \wedge^{r-l}(\nu^*\omega|_{F'})$, this implies that $\wedge^{r-l}\omega' \neq 0$, which is a contradiction.

Let F_p be the fiber of $F \to S_i$ over $p \in S_i$. Note that F_p is a normal crossing variety for a general point $p \in S_i$. Then, by the exact sequence, one can see that $H^0(F_p, \hat{\Omega}^1_{F_p}) \neq 0$ or $H^0(F_p, \hat{\Omega}^2_{F_p}) \neq 0$.

Take an $l + i$ dimensional complete intersection $H = H_1 \cap H_2 ... \cap H_{n-l-i}$ of very ample divisors of X passing through a general point $p \in S_i$. (When $l + i = n$, we put $H = X$.) Then H has only rational singularities. Put $\tilde{H} := f^{-1}(H)$ and put $g := f|_{\tilde{H}}$. Note that $g^{-1}(p) = F_p$ is a divisor of $\tilde{H}$ with normal crossings. By Lemma (1.2) $H^0(F_p, \hat{\Omega}^i_{F_p}) = 0$ for $i > 0$, which is a contradiction.

Corollary (1.5). *Let $\pi : \tilde{X} \to X$ be a birational projective morphism from a projective symplectic n-fold $\tilde{X}$ to a normal n-fold X. Then any π-exceptional divisor is mapped onto an $(n-2)$-dimensional subvariety of X by π.*

Proof. Take a π-exceptional divisor E. For some $i \geq 1$ we can take the E as an R_i in the proof of Proposition (1.4). Then $\dim \pi(E) \geq n - 2$.

Let X be a normal variety of dim n with canonical singularities. Let Σ be the singular locus of X. By [Re] there is a closed subset $\Sigma_0 \subset \Sigma$ such that each point of $\Sigma \setminus \Sigma_0$ has an analytic open neighborhood in X isomorphic to (rational double point) $\times(\mathbf{C}^{n-2}, 0)$. The locus Σ_0 is called the dissident locus. Generally we have $\dim \Sigma_0 \leq n - 3$. But, when X has a symplectic resolution, we have a stronger result.

Proposition (1.6). *Let $\pi : \tilde{X} \to X$ be a birational projective morphism from a projective symplectic n-fold $\tilde{X}$ to a normal n-fold X. Then X has only*

canonical singularities and its dissident locus Σ_0 has codimension at least 4 in X. Moreover, if $\Sigma \setminus \Sigma_0$ is non-empty, then $\Sigma \setminus \Sigma_0$ is a disjoint union of smooth varieties of dim $n-2$ with everywhere non-degenerate 2-forms.

Proof:

(1.6.1) Σ *has no (n-3)-dimensional irreducible components.*

We shall derive a contradiction by assuming that Σ has an $(n-3)$-dimensional irreducible component. Let $H := H_1 \cap H_2 \cap ... \cap H_{n-3}$ be a complete intersection of very ample divisors of X. The H intersects the $(n-3)$-dimensional component in finite points. Let $p \in H$ be one of such points. Let $H' := \pi^{-1}(H)$. Since there are no exceptional divisors of π lying on the $(n-3)$-dimensional component of Σ, $\pi|_H : H' \to H$ gives a small resolution of H around p. Pick an irreducible curve C from $\pi|_{H'}^{-1}(p)$. The C is isomorphic to $\mathbf{P}^1$, and its normal bundle $N_{C/H'}$ in H' is isomorphic to one of three vector bundles $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, $\mathcal{O}(-2) \oplus \mathcal{O}$ or $\mathcal{O}(-3) \oplus \mathcal{O}(1)$. Note that the Hilbert scheme $\mathrm{Hilb}(\tilde{X})$ has at most dimension $(n-3)$ because C can only move in $\tilde{X}$ along the $(n-3)$-dimensional component of Σ. However this contradicts Lemma (1.3).

(1.6.2). By (1.6.1) we only have to observe the irreducible components of Σ with dimension $n-2$. So we replace Σ by an irreducible component of Σ with dim $n-2$. We shall derive a contradiction by assuming that $\dim \Sigma_0 = n-3$.

Let $H := H_1 \cap H_2 \cap ... \cap H_{n-3}$ be a complete intersection of very ample divisors of X. Then $\tilde{H} := \pi^{-1}(H)$ is a crepant resolution of H. Set $\Lambda := \Sigma \cap H$ and $\Lambda_0 := \Sigma_0 \cap H$. Note that Λ_0 consists of finite points. Write $\tau : \tilde{H} \to H$ for the restriction $\pi|_{\tilde{H}}$ of π to $\tilde{H}$. Every fiber of τ has at most dimension one because, if some fibers are 2-dimensional, then there is a prime divisor of $\tilde{X}$ lying on Σ_0, which contradicts Corollary (1.5).

We shall show that Λ is a smooth curve and that $\mathrm{Exc}(\tau)$ is locally isomorphic to the product of Λ and a tree of $\mathbf{P}^1$'s. If so, then H must have rational double points of the same type along Λ and this is a contradiction. A contradiction will be deduced in several steps.

(i) Take a point $p_0 \in \Lambda_0$. We only have to argue locally around p_0. Since H has rational singularities and since $\tau^{-1}(p_0)$ is 1-dimensional, $\tau^{-1}(p_0)$ is a tree of $\mathbf{P}^1$'s. Let $C_1, ..., C_m$ be the irreducible components of $\tau^{-1}(p_0)$. Let us compute the normal bundle $N_{C_i/\tilde{H}}$. Take a sufficiently small open neighborhood U of $\cup C_i \subset \tilde{H}$. Since H has only rational singularities, we have $H^1(U, \mathcal{O}_U) = 0$. Let I_i be the defining ideal of C_i in U. Then, by the exact sequence $H^1(U, \mathcal{O}_U) \to H^1(C_i, \mathcal{O}_U/I_i^2) \to H^2(U, I_i^2) = 0$ we know that $H^1(C_i, \mathcal{O}_U/I_i^2) = 0$. By another exact sequence $H^0(C_i, \mathcal{O}_U/I_i^2) \to H^0(C_i, \mathcal{O}_U/I_i) (= \mathbf{C}) \to H^1(C_i, I_i/I_i^2) \to H^1(C_i, \mathcal{O}_U/I_i^2) = 0$, we know that $H^1(C_i, I_i/I_i^2) = 0$ because the first map is surjective. Therefore, $N_{C_i/\tilde{H}} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, $\mathcal{O}(-2) \oplus \mathcal{O}$ or $\mathcal{O}(-3) \oplus \mathcal{O}(1)$.

By Lemma (1.3), the Hilbert scheme $\mathrm{Hilb}(\tilde{X})$ is smooth of dimension $(n-2)$ at $[C_i]$. This fact tells us two things.

(i-a): *Each C_i moves inside $\tilde{H}$*; in fact, if C_i is rigid in $\tilde{H}$, then $\mathrm{Hilb}(\tilde{X})$ possibly has only (n-3) parameter at $[C_i]$ corresponding to a displacement of $C_i \subset \tilde{X}$ along Σ_0, which is a contradiction.

(i-b): *We have $N_{C_i/\tilde{H}} \cong \mathcal{O} \oplus \mathcal{O}(-2)$, in particular, $N_{C_i/\tilde{X}} \cong \mathcal{O}^{\oplus(n-2)} \oplus \mathcal{O}(-2)$.*

This fact can be proved by using Grothendieck's Hilbert scheme (cf. [Ko 1, Chap. I]): Let $\mathrm{Hilb}(\tilde{X}/X)$ be the relative Hilbert scheme for $\pi : \tilde{X} \to X$. Since C_i is contained in a fiber of π, $\mathrm{Hilb}(\tilde{X})$ coincides with $\mathrm{Hilb}(\tilde{X}/X)$ at $[C_i]$. Therefore $\mathrm{Hilb}(\tilde{X}/X)$ is smooth of dimension (n-2) at $[C_i]$. Moreover, the irreducible component of $\mathrm{Hilb}(\tilde{X}/X)$ containing $[C_i]$ dominates an $(n-2)$-dimensional irreducible component of Σ by the map $\mathrm{Hilb}(\tilde{X}/X) \to X$. By the universal property of the relative Hilbert scheme, we have $\mathrm{Hilb}(\tilde{H}/H) \cong \mathrm{Hilb}(\tilde{X}/X) \times_X H$, and hence $\mathrm{Hilb}(\tilde{H}/H)$ is smooth of dimension 1 at $[C_i]$ by Bertini theorem. Since $\mathrm{Hilb}(\tilde{H})$ coincides with $\mathrm{Hilb}(\tilde{H}/H)$ at $[C_i]$, this implies that $\mathrm{Hilb}(\tilde{H})$ is smooth of dimension 1 at $[C_i]$. Therefore we have $N_{C_i/\tilde{H}} \cong \mathcal{O} \oplus \mathcal{O}(-2)$.

(ii) We shall prove that Λ is irreducible around $p_0 \in \Lambda_0$. By (i-a) there are no flopping curves in $\mathrm{Exc}(\tau)$, hence τ is a *unique* crepant resolution of H. Therefore, we can construct τ locally around p_0 in the following manner. Let $\Lambda_1, ..., \Lambda_n$ be the irreducible components of Λ at p_0. Blow up H at first along the defining ideal I_1 of the reduced subscheme Λ_1 and take its normalization. We shall prove that τ is factorized by this composition of blow-up and normalization. We shall argue along the line of [Re 2, §2.12-15]. First note that H is a cDV point by [Re 1, Theorem (2.2)]. Let us view H as a total space of a flat family of surface rational double points over a disc Δ^1. The τ then can be viewed as a simultaneous (partial) resolution of this flat family. Let $F_1, ..., F_l$ be the irreducible components of $\mathrm{Exc}(\tau)$ which dominate Λ_1. There is a unique positive divisor $F = \Sigma a_i F_i$ such that F meets each general fiber $\tilde{H}_t$ $(t \in \Delta^1)$ in the sum of the Artin's fundamental cycles for the rational double points $H_t \cap \Lambda_1$. Since there are no rigid τ-exceptional curves, any τ-exceptional curve C moves along some Λ_i. If C moves along Λ_i with $i > 1$, then $(-F.C) = 0$. If C moves along Λ_1, then $(-F.C) \geq 0$ by the definition of F. Therefore, $-F$ is τ-nef divisor. At a general point of Λ_1, $\tau_*\mathcal{O}_{\tilde{H}}(-F)$ coincides with the defining ideal sheaf I_1 of the reduced subscheme Λ_1. Since every fiber of τ has dimension ≤ 1, $\tau_*\mathcal{O}_{\tilde{H}}(-F) \cong I_1$ (cf. [Re 2, (2.14)]. Since $-F$ is a τ-nef, τ-big divisor, the natural map $\tau^*\tau_*\mathcal{O}_{\tilde{H}}(-F) \to \mathcal{O}_{\tilde{H}}(-F)$ is surjective. Thus the ideal $\tau^{-1}I_1 \subset \mathcal{O}_{\tilde{H}}$ is invertible. Let H'_1 be the blowing up of H along I_1. Then τ is factorized as $\tilde{H} \to H'_1 \to H$.

Take an irreducible component of the singular locus of the resulting 3-fold which dominates Λ_1. Blow up the 3-fold along the defining ideal of this irreducible component with reduced structure, and then take the normalization. Repeating such procedure resolves singularities along general points of Λ_1. Denote by $\tau_1 : H_1 \to H$ the resulting 3-fold. Next take an irreducible component

of $\mathrm{Sing}(H_1)$ which dominates Λ_2. Blow up H_1 along the defining ideal of it and take the normalization. By repeating them, τ is finally decomposed as

$$\tilde{H} = H_n \overset{\tau_n}{\to} H_{n-1} \overset{\tau_{n-1}}{\to} ... \overset{\tau_1}{\to} H$$

We shall derive a contradiction by assuming $n \geq 2$. By (i-b) there is a smooth surface $E_i \subset \tilde{H}$ which has a $\mathbf{P}^1$-bundle structure containing C_i as a fiber. These surfaces E_i are mapped onto the *same* irreducible component of Λ by τ; indeed, if $C_i \cap C_j \neq \emptyset$ and $\tau(E_i) \neq \tau(E_j)$, then $E_i \cap E_j = \{\text{one point}\}$, which is a contradiction because both E_i and E_j are Cartier divisors of $\tilde{H}$. Moreover, $\tau(E_i) \neq \Lambda_n$. Indeed, suppose to the contrary. Then $C_1, ..., C_m$ are all contracted to a point by τ_n. At the same time, all exceptional divisors of τ lying on $\Lambda_1, ..., \Lambda_{n-1}$ are contracted to curves. By the construction of τ_i's, this is a contradiction.

On the other hand, the decomposition of τ explained above depends on the ordering of the irreducible components of Λ. Thus we have $\tau(E_i) \neq \Lambda_k$ for any $k \geq 1$, which is obviously a contradiction.

(iii) Let $E_i \subset \tilde{H}$ be a smooth divisor mentioned above. It has a $\mathbf{P}^1$-bundle structure containing C_i as a fiber. We shall prove

(iii-a): $\mathrm{Exc}(\tau)$ *is a divisor with simple normal crossings*;

(iii-b): $\mathrm{Exc}(\tau) = \bigcup_{1 \leq i \leq m} E_i$;

(iii-c): *If $C_i \cap C_j = \emptyset$, then $E_i \cap E_j = \emptyset$. If $C_i \cap C_j \neq \emptyset$, then $E_i \cap E_j$ is a section of at least one of the $\mathbf{P}^1$-bundles E_i and E_j*;

First we shall prove (iii-b). Since Λ is irreducible by (ii), $\tilde{H} = H_1$ and $\tau = \tau_1$ in the notation of (ii). The τ_1 is decomposed into blowing ups along irreducible reduced centers (followed by normalizations):

$$\tilde{H} \to ... \overset{\sigma_3}{\to} H^{(2)} \overset{\sigma_2}{\to} H^{(1)} \overset{\sigma_1}{\to} H$$

By the construction, $\mathrm{Exc}(\sigma_k)$ has a fibration over an irreducible curve whose general fiber is isomorphic to $\mathbf{P}^1$ or a reducible line pair. When a general fiber of the fibration is irreducible, the special fiber must be irreducible. Indeed, if the special fiber contains more than one irreducible component, then the proper transform of some of them to $\tilde{H}$ becomes a rigid rational curve, which contradicts (i-a). In this case $\mathrm{Exc}(\sigma_k)$ is irreducible.

When a general fiber of the fibration is reducible, the special fiber must have one or two irreducible components because, if it has more than two irreducible components, then the proper transform of some of them to $\tilde{H}$ becomes a rigid rational curve.

If the special fiber has exactly two irreducible components, then $\mathrm{Exc}(\sigma_k)$ has exactly two irreducible components.

We shall prove that if the special fiber is irreducible, then $\mathrm{Exc}(\sigma_k)$ is also irreducible. Suppose to the contrary. Denote by C the special fiber. Then

$\mathrm{Exc}(\sigma_k)$ has exactly two irreducible components F and F'. Each of them has a fibration over an irreducible curve, and the special fiber moves (as a 1-cycle on $H^{(k)}$) in both F and F'. Let $\tilde{F}$ (resp. $\tilde{F}'$) be the proper transform of F (resp. F') by $\tilde{\sigma} : \tilde{H} \to H^{(k)}$. The $\tilde{F}$ (resp. $\tilde{F}'$) has a fibration over an irreducible curve containing the proper transform $\tilde{C}$ of C in a special fiber. The special fiber has only one irreducible component $\tilde{C}$ because if it contains more, then $\tilde{C}$ is a rigid rational curve [2] and this contradicts (i-a).

Thus $\tilde{C}$ moves (as a 1-cycle) in both $\tilde{F}$ and $\tilde{F}'$. On the other hand, since $\tilde{C}$ coincides with one of C_i's, $\tilde{C}$ should move as fibers in only one smooth $\mathbf{P}^1$-bundle by (i-b). This is a contradiction.

As a consequence, we know that $\mathrm{Exc}(\tau)$ has exactly m irreducible components. Since E_i's are contained in $\mathrm{Exc}(\tau)$, (iii-b) holds.

We shall next prove (iii-c) and (iii-a). The first statement of (iii-c) is clear. Assume that $C_i \cap C_j \neq \emptyset$. Denote by $p_i : E_i \to \Delta^1$ (resp. $p_j : E_j \to \Delta^1$) the $\mathbf{P}^1$-bundle structure of E_i (resp. E_j) whose central fiber over $0 \in \Delta^1$ is C_i (resp. C_j). The intersection $E_i \cap E_j$ is multi-sections of p_i and p_j of degree n_i and n_j respectively. Suppose that $n_i > 1$ and $n_j > 1$.

Let $\mathcal{C}$ be the set of all irreducible curves on $\tilde{H}$ which are fibers of p_i or p_j. *For l, $l' \in \mathcal{C}$, we say l and l' are equivalent if there is a sequence of the elements of $\mathcal{C}$: $l_0 := l$, l_1, ..., $l_{k_0 - 1}$, $l_{k_0} := l'$ such that $l_k \cap l_{k+1} \neq \emptyset$ for any k.* This is an equivalence relation of $\mathcal{C}$.

Take a general fiber l^* of p_i and consider the set $\mathcal{C}(l^*)$ of all curves which are equivalent to l^*. Note that $\mathcal{C}(l^*)$ is a finite set consisting of smooth rational curves. Pick up an element $l \in \mathcal{C}(l^*)$ which is a fiber of p_i. Then there are at least n_i fibers of p_j which intersect l. Similarly, for any element $m \in \mathcal{C}(l^*)$ which is a fiber of p_j, there are at least n_j fibers of p_i which intersect m. This implies that $\mathcal{C}(l^*)$ is *not* a tree of $\mathbf{P}^1$'s.

On the other hand, $\mathcal{C}(l^*)$ is contained in a fiber of $\tau : \tilde{H} \to H$, which is a contradiction. Therefore, $n_i = 1$ or $n_j = 1$. One can assume that $n_i = 1$. In this case, $E_i \cap E_j$ is a section of p_i, and E_j intersects E_i with multiplicity one along $E_i \cap E_j$ because, if not, then it contradicts the fact that each fiber of τ is a tree of $\mathbf{P}^1$'s. Since there are no triple points in $E_1 \cup E_2 \cup ... \cup E_m$, (iii-a) and (iii-c) hold.

(iv) We shall prove that $E_i \cap E_j$ is sections of *both* $\mathbf{P}^1$-bundles E_i and E_j in (iii-c). If this is proved, then $\mathrm{Exc}(\tau)$ is locally the product of a one-dimensional disk Δ^1 and a tree of $\mathbf{P}^1$'s.

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[2] By Theorem (2.2) from [Re 1], we know that $H^{(k-1)}$ has only cDV sinularities. Put $p := \sigma_k(C)$. The germ $(H^{(k-1)}, p)$ is then isomorphic to a hypersurface singularity $x^2 + f(y, z, w) = 0$ with $deg(f) \geq 3$. There is an involution ι of $(H^{(k-1)}, p)$ defined by $x \to -x$, $y \to y$, $z \to z$ and $w \to w$. Since $\tilde{H}$ is a unique crepant resolution of $H^{(k-1)}$, the ι lifts to an involution $\tilde{\iota}$ of $(\tilde{H}, \tilde{\sigma}^{-1}(C))$. By $\tilde{\iota}$, $\tilde{F}$ and $\tilde{F}'$ are interchanged. Therefore, if the special fiber for $\tilde{F}$ is reducible, then the special fiber for $\tilde{F}'$ is also reducible. Any $\tilde{\sigma}$-exceptional divisor does not contain $\tilde{C}$. Since $\tilde{C}$ does not move in $\tilde{F}$ or $\tilde{F}'$, $\tilde{C}$ must be rigid in $\tilde{H}$.

Therefore, Λ is smooth at $p_0 \in \Lambda_0$ [3].

Assume that $E_i \cap E_j$ is a section of $p_i : E_i \to \Delta^1$, but is a multiple section of $p_j : E_j \to \Delta^1$ of degree > 1. Let $Q \in C_i \cap C_j$. Since $N_{C_j/\tilde{X}} \cong \mathcal{O}^{\oplus(n-2)} \oplus \mathcal{O}(-2)$ and since $\mathrm{Hilb}(\tilde{X})$ is smooth at $[C_j]$, we can take an (n-2) dimensional subvariety D of $\tilde{X}$ (at least locally around C_j) such that (1): D has a $\mathbf{P}^1$-bundle structure over an (n-3)-dimensional disc Δ^{n-3} which contains C_j as a central fiber, (2): D meets $\tilde{H}$ in C_j normally. Take (n-3) coordinate axis Δ_k $(1 \leq k \leq n-3)$ of Δ^{n-3}. By pulling back the $\mathbf{P}^1$-bundle $D \to \Delta^{n-3}$ by $\Delta_k \to \Delta^{n-3}$, we have a smooth surface D_k which has a $\mathbf{P}^1$-bundle structure over Δ_k.

We take local coordinates $(x, y, z_1, z_2, ..., z_{n-3}, t)$ at $Q \in \tilde{X}$ in such a way that

(a) $\mathbf{C} < \partial/\partial x >= T_{C_i,Q}$;

(b) $\mathbf{C} < \partial/\partial y >= T_{C_j,Q}$;

(c) $\mathbf{C} < \partial/\partial x, \partial/\partial y >= T_{E_i,Q}$;

(d) $\mathbf{C} < \partial/\partial y, \partial/\partial t >= T_{E_j,Q}$;

(e) $\mathbf{C} < \partial/\partial y, \partial/\partial z_k >= T_{D_k,Q}$, k = 1,2, ..., n-3.

Let ω be a non-degenerate 2-form on $\tilde{X}$. At Q, ω_Q can be written as a linear combination of $dx \wedge dy$, $dx \wedge dz_k$, $dx \wedge dt$, $dy \wedge dz_k$, $dy \wedge dt$, $dz_k \wedge dz_l$ and $dz_k \wedge dt$.

Since E_i, E_j and D_k $(1 \leq k \leq n-3)$ are all $\mathbf{P}^1$-bundles over smooth curves, $\omega|_{E_i} = \omega|_{E_j} = \omega|_{D_k} = 0$. In particular, the terms $dx \wedge dy$, $dy \wedge dz_k$, and $dy \wedge dt$ never appear in ω_Q. By definition, $\wedge^{n/2}\omega_Q \neq 0$, but this is a contradiction.

(1.6.3). *there is a non-degenerate 2-form on* $\Sigma \setminus \Sigma_0$.

By (1.6.2), $\Sigma \setminus \Sigma_0$ is a smooth (n-2)-dimensional subvariety. Let $E^0 := \pi^{-1}(\Sigma \setminus \Sigma_0)$. E^0 is a $\mathbf{P}^1$-tree bundle over $\Sigma \setminus \Sigma_0$. The non-degenerate 2-form ω on $\tilde{X}$ is restricted to a non-zero 2-form ω' on E^0. The ω' must be the pull back of a 2-form on $\Sigma \setminus \Sigma_0$. Since $\wedge^{n/2-1}\omega'$ does not vanish on the smooth part of E^0 (cf. Proof of (1.4)), this 2-form on $\Sigma \setminus \Sigma_0$ should be non-degenerate.

Examples (1.7). **(i)** Let S be a projective K3 surface containing a (-2)-curve C. Let $S \to \overline{S}$ be the birational contraction map sending C to a point $p \in \overline{S}$. Let $\tilde{X} := \mathrm{Hilb}^2(S)$ be the Hilbert scheme parametrizing lenth 2 points on S. Note that $\tilde{X}$ is a symplectic 4-fold obtained as a resolution of the symmetric

[3] The proof is as follows. Let $\Sigma a_i E_i$ be the fundamental cycle in the sense of Artin. Let i_1 be an index which attain the maximal coefficient in the cycle. Consider a $\mathbf{Q}$-divisor $G := \Sigma(a_i/a_{i_1})E_i$. Define $\lceil -G \rceil := \Sigma \lceil -a_i/a_{i_1} \rceil E_i$, where $\lceil -a_i/a_{i_1} \rceil$ denote the smallest integer r satisfying $r \geq -a_i/a_{i_1}$. By definition, we have $\lceil -G \rceil = -\Sigma_{1 \leq l \leq k} E_{i_l}$, where i_1, i_2, ..., i_k run through all indices which attain the maximal coefficient in the cycle. Since $-G$ is τ-nef, $R^1\tau_*\mathcal{O}_{\tilde{H}}(\lceil -G \rceil) = 0$ by Kawamata-Viehweg vanishing theorem. For simplicity, we write $E := \Sigma_{1 \leq i \leq m} E_i$ and $E' := E + \lceil -G \rceil$. By the exact sequence $\tau_*\mathcal{O}_{E'}(\lceil -G \rceil) \to R^1\tau_*\mathcal{O}_{\tilde{H}}(-E) \to R^1\tau_*\mathcal{O}_{\tilde{H}}(\lceil -G \rceil)$, we see that $R^1\tau_*\mathcal{O}_{\tilde{H}}(-E) = 0$ because the 1-st term vanishes in the sequence. The natural map $\tau_*\mathcal{O}_{\tilde{H}} \to \tau_*\mathcal{O}_E$ is therefore a surjection, which implies that Λ is a normal curve, hence is smooth.

product $\mathrm{Sym}^2(S) := S \times S/\mathbf{Z}_2$ of S (cf. [Fu]). Let X be the symmetric product $\mathrm{Sym}^2(\overline{S})$ of $\overline{S}$. Then there is a birational projective morphism $\pi : \tilde{X} \to X$. The singular locus Σ of X consists of two irreducible components $\Sigma^{(1)}$ and $\Sigma^{(2)}$, where both of them are isomorphic to $\overline{S}$ and $\Sigma^{(1)} \cap \Sigma^{(2)} = \{(p,p)\}$. We can take the 0-dimensional subvariety $\{(p,p)\}$ as the Σ_0 in (1.6). X has A_1 singularities along Σ except at $\{(p,p)\}$. The fiber $\pi^{-1}((p,p))$ has two irreducible components which are isomorphic to $\mathbf{P}^2$ and the Hirzebruch surface F_1.

(ii) This is an example of a small birational contraction map $\pi : \tilde{X} \to X$ where its flop (= elementary transformation) does not preserve the projectivity (and also Kaehlerity). In particular, π is not a projective morphism.

Let $S \to \mathbf{P}^1$ be an elliptic, projective K3 surface which has two type-I_3 singular fibers (cycle of three smooth rational curves). Denote by $C = C_1 + C_2 + C_3$ and by $D_1 + D_2 + D_3$ these singular fibers. Let $\tilde{X} := \mathrm{Hilb}^2(S)$ be the Hilbert scheme parametrizing length 2 points on S. There is a birational projective morphism $\mu : \tilde{X} \to \mathrm{Sym}^2(S)$. Let F_i (resp. G_i) be the proper transform of $\mathrm{Sym}^2(C_i) \subset \mathrm{Sym}^2(S)$ (resp. $\mathrm{Sym}^2(D_i) \subset \mathrm{Sym}^2(S)$) by μ. The F_i's and G_i's are mutually disjoint and they are isomorphic to $\mathbf{P^2}$. Sinse $N_{F_i/\tilde{X}} \cong \Omega^1_{\mathbf{P^2}}$, there exists a (not necessarily projective) birational map $\pi : \tilde{X} \to X$ which contracts F_1, F_2 and F_3 to points. Let $\pi^+ : \tilde{X}^+ \to X$ be the flop of π. $\tilde{X}^+$ is obtained by elementary transformations along F_i's.

We shall prove that $\tilde{X}^+$ is non-projective (hence is non-Kaehler because $\tilde{X}^+$ is a Moishezon variety). Let l_i be a line on F_i and m_i a line on G_i. Then $l_1+l_2+l_3$ is numerically equivalent to $m_1+m_2+m_3$ as an algebraic 1-cycle on $\tilde{X}$. In fact, let $L \in \mathrm{Pic}(\tilde{\mathrm{X}})$. Then $L \cong \mathcal{O}_{\tilde{X}}(aE) \otimes \mu^* M$, where E is the μ-exceptional divisor, $M \in \mathrm{Pic}(\mathrm{Sym}^2(\mathrm{S}))$ and $a \in \mathbf{Z}$. Let $p : S \times S \to \mathrm{Sym}^2(S)$ be the natural Galois cover with Galois group $G = \mathbf{Z}/\mathbf{2Z}$. Let $q_j : S \times S \to S$ be the j-th projection ($j = 1, 2$). Since $H^1(S, \mathcal{O}_S) = 0$, we can write $p^*M = q_1^* M_1 \otimes q_2^* M_2$ with M_1, $M_2 \in \mathrm{Pic(S)}$. Since p^*M is G-invariant, $M_1 = M_2$. Therefore

$$(L.l_i)_{\tilde{X}} = (aE.l_i)_{\tilde{X}} + (M_1.C_i)_S = a + (M_1.C_i)_S$$

$$(L.m_i)_{\tilde{X}} = (aE.m_i)_{\tilde{X}} + (M_1.D_i)_S = a + (M_1.D_i)_S$$

As $C_1 + C_2 + C_3$ is linearly equivalent to $D_1 + D_2 + D_3$ on S, it follows that $(L.l_1 + l_2 + l_3)_{\tilde{X}} = (L.m_1 + m_2 + m_3)_{\tilde{X}}$. Therefore $l_1 + l_2 + l_3$ is numerically equivalent to $m_1 + m_2 + m_3$. We shall derive a contradiction assuming that $\tilde{X}^+$ is projective. Let H be an ample divisor of $\tilde{X}^+$. Let $H' \in \mathrm{Pic}(\tilde{\mathrm{X}})$ be the strict transform of H by the birational map $\phi : \tilde{X}^+ - - \to \tilde{X}$. By definition of an elementary transformation H' is negative along each F_i. H' is positive along each G_i because ϕ is an isomorphism around each G_i. In particular, $(H'.l_i) < 0$, and $(H'.m_i) > 0$. Hence $(H'.l_1 + l_2 + l_3) < 0$ and $(H'.m_1 + m_2 + m_3) > 0$, which is a contradiction because $l_1 + l_2 + l_3$ is numerically equivalent to $m_1 + m_2 + m_3$. Finally note that $\pi : \tilde{X} \to X$ is not a projective morphism because if so, then X

is projective and hence $\tilde{X}^+ = \mathrm{Proj}_X(\oplus_m \pi_* \mathcal{O}_{\tilde{X}}(mH'))$ is also projective, which is a contradiction.

(iii) Let A be an Abelian surface, and let $\mathrm{Hilb}^3(A)$ be the Hilbert scheme parametrizing length 3 points on A. The Albanese map $Alb : \mathrm{Hilb}^3(A) \to A$ factors through the symmetric product $\mathrm{Sym}^3(A) := A \times A \times A/S_3$ as $\mathrm{Hilb}^3(A) \to \mathrm{Sym}^3(A) \xrightarrow{f} A$. For $(x, y, z) \in \mathrm{Sym}^3(A)$, $f(x, y, z) = x + y + z \in A$. Take the origin $0 \in A$, and set $\tilde{X} := \mathrm{Alb}^{-1}(0)$ and $X := f^{-1}(0)$. The $\tilde{X}$ is called a symplectic manifold of Kummer type, and is often denoted by $\mathrm{Kum}^2(A)$. There is a birational projective morphism $\pi : \tilde{X} \to X$. Note that $\tilde{X}$ is a symplectic 4-fold (cf. [Be]). The singular locus Σ of X is isomorphic to A, and X has A_1 singularities along Σ except at 81 points $\{p_i\}$ $(1 \leq i \leq 81)$. The fiber $\pi^{-1}(p_i)$ is homeomorphic to the normal surface $\overline{F}_3$ obtained from the Hirzebruch surface F_3 by contracting (-3)-curve to a point (cf. [Br]). We can take these 81 points as the Σ_0 in (1.6)

(iv) Let V be a smooth cubic 4-fold in $\mathbf{P^5}$. Let Y be the Hilbert scheme parametrizing lines contained in V, which is called classically a Fano scheme. Then Y is a symplectic manifold of dimension 4. Moreover, Y is deformation equivalent to $\mathrm{Hilb}^2(S)$ which parametrizes length 2 points of a K3 surface S ([B-D]).

We choose a cubic 4-fold V defined by the equation $f(T_0, T_2, T_4) + g(T_1, T_3, T_5) = 0$ where T_i's are homogenous coordinates of $\mathbf{P^5}$. The cyclic group $G = \mathbf{Z}/\mathbf{3Z}$ acts on $\mathbf{P^5}$ by $T_0 \to T_0$, $T_1 \to \zeta T_1$, $T_2 \to T_2$, $T_3 \to \zeta T_3$, $T_4 \to T_4$ and $T_5 \to \zeta T_5$, where ζ is a primitive 3 root of 1. G acts also on V, and hence naturally on Y. By using a G equivariant isomorphism $H^1(V, \Omega^3_V) \cong H^0(Y, \Omega^2_Y)$ ([B-D]), we know that this G action preserves a symplectic 2-form on Y.

Let us observe the G action on Y in more detail. We denote by P and P' the projective planes defined by $T_1 = T_3 = T_5 = 0$ and $T_0 = T_2 = T_4 = 0$ respectively. Define C to be the cubic curve on P defined by $f(T_0, T_2, T_4) = 0$ and define D to be the cubic curve on P' defined by $g(T_1, T_3, T_5) = 0$. The fixed locus F of the G action on Y is the set of lines which join two points $p \in C$ and $q \in D$. Hence $F \cong C \times D$.

We put $X := Y/G$. Then X is a symplectic V-manifold of dim 4 and its singular locus Σ is isomorphic to F. X has A_2 singularities along Σ. Then we can take a symplectic resolution $\pi : \tilde{X} \to X$. It is checked that $\tilde{X}$ is birationally equivalent to $\mathrm{Kum}^2(C \times D)$.

(1.8) *The Structure of the Singular Locus*:

Let $\pi : \tilde{X} \to X$, Σ and Σ_0 be the same as (1.6). Set $U := X \setminus \Sigma_0$, $\tilde{U} = \pi^{-1}(U)$, $\pi_U := \pi|_{\tilde{U}}$ and $D_U := \mathrm{Exc}(\pi_U)$.

We shall study the structure of the sheaf $T^1_U := \underline{Ext}^1(\Omega^1{}_U, \mathcal{O}_U)$. Note that T^1_U has support on $\Sigma \setminus \Sigma_0$.

Let $D_1, ..., D_m$ be the irreducible components of D_U. D_U is a divisor with normally crossing double points. Write $D_{i,j}$ for $D_i \cap D_j$.

We shall describe the possible configurations of D_i's over each connected component of $\Sigma \setminus \Sigma_0$. We shall assume, for simplicity, that $\Sigma \setminus \Sigma_0$ is connected.

Let $\pi_1 : U_1 \to U$ be the blowing-up with reduced center $\Sigma \setminus \Sigma_0$. Then we have:

(A_1): If U has A_1 singularities along $\Sigma \setminus \Sigma_0$, then $\mathrm{Exc}(\pi_1)$ is a $\mathbf{P}^1$-bundle over $\Sigma \setminus \Sigma_0$. In this case $\tilde{U} = U_1$ and $\pi_U = \pi_1$.

(A_n), $(n \geq 2)$: If U has A_n singularities along $\Sigma \setminus \Sigma_0$, then $\mathrm{Exc}(\pi_1)$ has a fibration over $\Sigma \setminus \Sigma_0$ whose general fiber is a (reducible) pair of two lines. The U_1 has A_{n-2} singularities along the double points of $\mathrm{Exc}(\pi_1)$. Note that $\mathrm{Exc}(\pi_1)$ is possibly *irreducible* when $\Sigma \setminus \Sigma_0$ has non-trivial fundamental group.

(D_4): If U has D_4 singularities along $\Sigma \setminus \Sigma_0$, then $\mathrm{Exc}(\pi_1)$ is a $\mathbf{P}^1$-bundle over $\Sigma \setminus \Sigma_0$. There is an etale multiple section $S \subset \mathrm{Exc}(\pi_1)$ of degree 3 such that U_1 has A_1 singularities along S. The S possibly has one, two or three connected components.

(D_n) $(n \geq 5)$: If U has D_n $(n \geq 5)$ singularities along $\Sigma \setminus \Sigma_0$, then $\mathrm{Exc}(\pi_1)$ is a $\mathbf{P}^1$-bundle over $\Sigma \setminus \Sigma_0$. There are two disjoint sections S_1 and S_2 such that U_1 has A_1 singularities along S_1 and has D_{n-2} (A_3 when $n = 5$) singularities along S_2.

(E_6): If U has E_6 singularities along $\Sigma \setminus \Sigma_0$, then $\mathrm{Exc}(\pi_1)$ is a $\mathbf{P}^1$-bundle over $\Sigma \setminus \Sigma_0$. There is a section S along which U_1 has A_5 singularities.

(E_7): If U has E_7 singularities along $\Sigma \setminus \Sigma_0$, then $\mathrm{Exc}(\pi_1)$ is a $\mathbf{P}^1$-bundle over $\Sigma \setminus \Sigma_0$. There is a section S along which U_1 has D_6 singularities.

(E_8): If U has E_8 singularities along $\Sigma \setminus \Sigma_0$, then $\mathrm{Exc}(\pi_1)$ is a $\mathbf{P}^1$-bundle over $\Sigma \setminus \Sigma_0$. There is a section S along which U_1 has E_7 singularities.

Successive blowing ups with singular locus give us the (minimal) resolution $\pi_U : \tilde{U} = U_k \to U_{k-1} \to ... \to U_1 \to U$. Note that each step is essentially the same as one of π_1's described above. We can explicitly check that the number m of irreducible components of D_U are as follows according to the type of singularities of U.

$(A_n) => m = n$ or $n - [n/2]$.
$(D_4) => m = 4, 3$ or 2.
(D_n) $(n \geq 5) => m = n$ or $n - 1$.
$(E_6) => m = 6$ or 4.
$(E_7) => m = 7$.
$(E_8) => m = 8$.

A pair of the type of singularities of U and the number m is called a type of U. For example, if U has A_n singularities along $\Sigma \setminus \Sigma_0$ and $m = n - [n/2]$, then U is of type $(A_n, n - [n/2])$.

Let I be the defining ideal of the reduced subscheme $\Sigma \setminus \Sigma_0$ of U. Denote by $\Sigma^{(n)}$ the subscheme of U defined by I^{n+1}. Then there is a sequence of subschemes supported at $\Sigma \setminus \Sigma_0$: $\Sigma^{(0)} \subset \Sigma^{(1)} \subset \Sigma^{(2)} \subset ....$

Lemma (1.9). *The sheaf T_U^1 is described as a sequence of extensions according to the type of U in the following way:*

(A_1): $T_U^1 \cong T_U^1|_{\Sigma^{(0)}} \cong \mathcal{O}_{\Sigma^{(0)}}$

(A_n), $(n \geq 2)$:

$$0 \to L \to T_U^1|_{\Sigma^{(1)}} \to \mathcal{O}_{\Sigma^{(0)}} \to 0$$

$$0 \to L^{\otimes 2} \to T_U^1|_{\Sigma^{(2)}} \to T_U^1|_{\Sigma^{(1)}} \to 0$$

$$\vdots$$

$$0 \to L^{\otimes n-1} \to T_U^1|_{\Sigma^{(n-1)}} \to T_U^1|_{\Sigma^{(n-2)}} \to 0$$

$$T_U^1 \cong T_U^1|_{\Sigma^{(n-1)}}$$

where L is non-trivial line bundle on $\Sigma^{(0)}$ with $L^{\otimes 2} \cong \mathcal{O}_{\Sigma^{(0)}}$ if U is of type $(A_n, n - [n/2])$, and where $L \cong \mathcal{O}_{\Sigma^{(0)}}$ if U is of type (A_n, n).

(D_4):

$$0 \to E \to T_U^1|_{\Sigma^{(1)}} \to \mathcal{O}_{\Sigma^{(0)}} \to 0$$

$$0 \to \mathcal{O}_{\Sigma^{(0)}} \to T_U^1|_{\Sigma^{(2)}} \to T_U^1|_{\Sigma^{(1)}} \to 0$$

$$T_U^1 \cong T_U^1|_{\Sigma^{(2)}}$$

where E is a vector bundle of rank 2 with $h^0(E) = 0$ if U is of type $(D_4, 2)$, where $E \cong \mathcal{O}_{\Sigma^{(0)}} \oplus L$ with a non-trivial line bundle L, $L^{\otimes 2} \cong \mathcal{O}_{\Sigma^{(0)}}$ if U is of type $(D_4, 3)$, and where $E \cong \mathcal{O}_{\Sigma^{(0)}}^{\oplus 2}$ if U is of type $(D_4, 4)$.

(D_n) $(n \geq 5)$:

$$0 \to \mathcal{O}_{\Sigma^{(0)}} \oplus L \to T_U^1|_{\Sigma^{(1)}} \to \mathcal{O}_{\Sigma^{(0)}} \to 0$$

$$0 \to \mathcal{O}_{\Sigma^{(0)}} \to T_U^1|_{\Sigma^{(2)}} \to T_U^1|_{\Sigma^{(1)}} \to 0$$

$$\vdots$$

$$0 \to \mathcal{O}_{\Sigma^{(0)}} \to T_U^1|_{\Sigma^{(n-2)}} \to T_U^1|_{\Sigma^{(n-3)}} \to 0$$

$$T_U^1 \cong T_U^1|_{\Sigma^{(n-2)}}$$

where L is a non-trivial line bundle with $L^{\otimes 2} \cong \mathcal{O}_{\Sigma^{(0)}}$ if U is of type $(D_n, n-1)$ and where $L \cong \mathcal{O}_{\Sigma^{(0)}}$ if U is of type (D_n, n).

(E_6):

$$0 \to E_1 \to T^1_U|_{\Sigma^{(1)}} \to \mathcal{O}_{\Sigma^{(0)}} \to 0$$

$$0 \to E_2 \to T^1_U|_{\Sigma^{(2)}} \to T^1_U|_{\Sigma^{(1)}} \to 0$$

$$0 \to \mathcal{O}_{\Sigma^{(0)}} \to T^1_U|_{\Sigma^{(3)}} \to T^1_U|_{\Sigma^{(2)}} \to 0$$

$$T^1_U \cong T^1_U|_{\Sigma^{(3)}}$$

where E_1 and E_2 are vector bundles on $\Sigma^{(0)}$ of rank 2 obtained as the following extensions

$$0 \to \mathcal{O}_{\Sigma^{(0)}} \to E_1 \to L \to 0$$

$$0 \to L \to E_2 \to \mathcal{O}_{\Sigma^{(0)}} \to 0$$

where L is a non-trivial line bundle with $L^{\otimes 2} \cong \mathcal{O}_{\Sigma^{(0)}}$ if U is of type $(E_6, 4)$, and where $L \cong \mathcal{O}_{\Sigma^{(0)}}$ if U is of type $(E_6, 6)$.

$(E_7, 7)$, $(E_8, 8)$:

$$0 \to E_1 \to T^1_U|_{\Sigma^{(1)}} \to \mathcal{O}_{\Sigma^{(0)}} \to 0$$

$$0 \to E_2 \to T^1_U|_{\Sigma^{(2)}} \to T^1_U|_{\Sigma^{(1)}} \to 0$$

$$0 \to E_3 \to T^1_U|_{\Sigma^{(3)}} \to T^1_U|_{\Sigma^{(2)}} \to 0$$

$$0 \to \mathcal{O}_{\Sigma^{(0)}} \to T^1_U|_{\Sigma^{(4)}} \to T^1_U|_{\Sigma^{(3)}} \to 0$$

$$T^1_U \cong T^1_U|_{\Sigma^{(4)}}$$

where E_1 and E_2 are vector bundles of rank 2 obtained as extensions of trivial line bundles : $0 \to \mathcal{O}_{\Sigma^{(0)}} \to E_i \to \mathcal{O}_{\Sigma^{(0)}} \to 0$. Moreover, if U is of type $(E_7, 7)$, then $E_3 \cong \mathcal{O}_{\Sigma^{(0)}}$, and if U is of type $(E_8, 8)$, then E_3 is a vector bundle of rank 2 obtained as an extension of trivial line bundles : $0 \to \mathcal{O}_{\Sigma^{(0)}} \to E_3 \to \mathcal{O}_{\Sigma^{(0)}} \to 0$.

The proof of this lemma is omitted; I will write it elsewhere. Note that we need two facts to prove it: $\omega_U \cong \mathcal{O}_U$ and $\omega_{\Sigma(0)} \cong \mathcal{O}_{\Sigma(0)}$. An important collorary of (1.9) is the following:

Corollary (1.10).

$$h^0(U, T^1_U) \leq m.$$

Proof. Note that, by Proposition (1.6), $\Sigma \setminus \Sigma_0$ is compactified to a proper normal variety $\overline{\Sigma}$ by adding codimension 2 points. We see that $h^0(U, T^1_U) \leq m$ by the extensions in (1.9) in any case.

We shall finally state related results to (1.4), which will not be used in the later. Proposition (1.4) is also available if we replace $\pi : \tilde{X} \to X$ by a projective symplectic resolution $\phi : \tilde{V} \to V$ of the germ of a rational Gorenstein singularity $0 \in V$ of even dimension n. When V is an isolated singularity, we have the following.

Proposition (1.11). *Let $\phi : \tilde{V} \to V$ be a projective symplectic resolution of the germ of an isolated rational Gorenstein singularity $0 \in V$ of dimension $n \geq 4$, that is, $\tilde{V}$ admits a non-degenerate holomorphic 2-form. Then every irreducible component of the exceptional locus $\mathrm{Exc}(\phi)$ is Lagrangian.*

Proof. First we shall prove:

Lemma (1.12). *Let $\phi : \tilde{V} \to V$ be a projective, crepant resolution of an isolated rational Gorenstein singularity of even dimension n. Then every irreducible component of $E := \mathrm{Exc}(\phi)$ has dimension $\geq n/2$.*

Proof of (1.12). Take an effective divisor Δ of $\tilde{X}$ in such a way that $-\Delta$ is ϕ-ample. If $\epsilon > 0$ is a sufficiently small rational number, then $(\tilde{X}, \epsilon\Delta)$ is log terminal in the sense of [Ka 2]. Since $K_{\tilde{X}} \sim 0$, every irreducible component of E is covered by a family of rational curves by [Ka 2, Theorem 1].

We shall now use the terminology in [Ko, Chap. IV]. Let E_i be an irreducible component of E. Let $\mathrm{Hom}_{bir}(\mathbf{P}^1, E_i)$ be the Hom scheme parametrizing the morphisms from $\mathbf{P}^1$ to E_i which are birational onto their images. Let $\mathrm{Hom}^n_{bir}(\mathbf{P}^1, E_i)$ be the normalization of $\mathrm{Hom}_{bir}(\mathbf{P}^1, E_i)$. By [Ko 1, Chap.IV, Theorem 2.4], there is an irreducible component W_i of $\mathrm{Hom}^n_{bir}(\mathbf{P}^1, E_i)$ such that W_i is a *generically unsplit family* of morphisms and such that $\overline{\mathrm{Locus}(W_i)} = E_i$, where $\mathrm{Locus}(W_i)$ is the locus where the images of the morphisms in W_i sweep out and $\overline{\mathrm{Locus}(W_i)}$ is its closure.

Let $[f] \in W_i$ be a general point. Then W_i is also an irreducible component of $\mathrm{Hom}^n_{bir}(\mathbf{P}^1, \tilde{V})$ at $[f]$. We know that $\dim_{[f]} W_i \geq \chi(\mathbf{P}^1, f^*\Theta_{\tilde{V}}) = n$. For the generically unsplit family W_i of morphisms, we can estimate $\mathrm{codim}(\mathrm{Locus}(W_i) \subset \tilde{V})$ (cf. [Io, Theorem (0.4), Ko, Chap.IV, Proposition (2.5)]). The result is

$$\mathrm{codim}(\mathrm{Locus}(W_i) \subset \tilde{V}) \leq n/2 + 1/2.$$

Note here that n is even. Since $\overline{\mathrm{Locus}(W_i)} = E_i$, we have $\dim E_i \geq n/2$. Q.E.D.

Proof of Proposition (1.11) continued. By combining Lemma (1.12) with Proposition (1.4) we have $\dim E_i = n/2$ for each irreducible component E_i of E. Let us prove that E_i are all Lagrangian. Let ω be a non-degenerate 2-form on $\tilde{V}$. Assume that $\omega|_{E_i} \neq 0$ for some E_i. Take a birational projective morphism $\nu : Y \to \tilde{V}$ in such a way that $\nu^{-1}(E)$ becomes a divisor of a smooth n-fold Y with normal crossings. Write $g = \phi \circ \nu$ for short. Then we have $H^0(g^{-1}(0), \hat{\Omega}^2_{g^{-1}(0)}) \neq 0$. This contradicts Lemma (1.2). Q.E.D.

2. Deformation theory

We shall review some generalities of deformation theory. For a compact complex space X we denote by $\mathrm{Def}(X)$ the Kuranishi space of X. By definition, there is a reference point $0 \in \mathrm{Def}(X)$ and there is a semi-universal flat deformation $f : \mathcal{X} \to \mathrm{Def}(\mathcal{X})$ of X with $f^{-1}(0) = X$. When X is reduced, the tangent space $T_{\mathrm{Def}(X),0}$ is canonically isomorphic to $\mathrm{Ext}^1(\Omega^1_{\mathrm{X}}, \mathcal{O}_{\mathrm{X}})$. We abbreviate this space by $\mathbf{T}^1_X$.

Let $\pi : \tilde{X} \to X$ be a proper bimeromorphic map of compact complex spaces. Assume that $R^1\pi_*\mathcal{O}_{\tilde{X}} = 0$ and $\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$. Then there is a natural map $\mathrm{Def}(\tilde{X}) \to \mathrm{Def}(X)$ (cf. [Ko-Mo, 11.4]). This map naturally induces a map $\pi_* : \mathbf{T}^1_{\tilde{X}} \to \mathbf{T}^1_X$. Assume that $\tilde{X}$ and X are both reduced. Then π_* is obtained as follows.

Let $0 \to \mathcal{O}_{\tilde{X}} \to F \to \Omega^1_{\tilde{X}} \to 0$ be the extension corresponding to an element of $\mathbf{T}^1_{\tilde{X}}$. Operate π_* on this sequence. Then we have an exact sequence $0 \to \mathcal{O}_X \to \pi_*F \to \pi_*\Omega^1_{\tilde{X}} \to 0$ because $R^1\pi_*\mathcal{O}_{\tilde{X}} = 0$ and $\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$. This extension gives an element of $\mathrm{Ext}^1(\pi_*\Omega^1_{\tilde{X}}, \mathcal{O}_X)$. By the natural map $\Omega^1_X \to \pi_*\Omega^1_{\tilde{X}}$, we obtain an element of $\mathbf{T}^1_X$.

In the remainder of this section, $\tilde{X}$ is a smooth projective symplectic n-fold with $n \geq 4$ and $\pi : \tilde{X} \to X$ is a birational projective morphism from $\tilde{X}$ to a normal n-fold X. We shall use the same notation as (1.6). Set $U := X \setminus \Sigma_0$.

Proposition (2.1). *There is a commutaive diagram*

$$\begin{array}{ccc} H^1(\tilde{X}, \Theta_{\tilde{X}}) & \longrightarrow & H^1(\pi^{-1}(U), \Theta_{\pi^{-1}(U)}) \\ \downarrow & & \downarrow \\ \mathbf{T}^1_X & \longrightarrow & \mathbf{T}^1_U \end{array} \tag{3}$$

and the horizontal maps are both isomorphisms.

Proof. Since X has rational singularities, $R^1\pi_*\mathcal{O}_{\tilde{X}} = 0$ and hence we have vertical maps by [Bi, Wa]. The horizontal map on the second row is an isomorphism by the argument of [Na 4, Propositions (1.1), (1.2)] or [Ko-Mo, (12.5.6)].

We only have to prove that the horizontal map on the first row is an isomorphism.

Since we will treat non-compact varieties, we shall fix the notation here. Assume that a complex space W admits a structure of an algebraic scheme W^{alg} over $\mathbf{C}$. Let F be a coherent analytic sheaf F on W which comes from an algebraic coherent sheaf F^{alg} on W^{alg}. Then we write $H^*(W, F^{alg})$ for $H^*(W^{alg}, F^{alg})$ by abuse of notation. Note that there is a natural map $H^*(W, F^{alg}) \to H^*(W, F)$.

We can see that the map $H^1(\pi^{-1}(U), \Theta^{alg}_{\pi^{-1}(U)}) \to H^1(\pi^{-1}(U), \Theta_{\pi^{-1}(U)})$ is an isomorphism (cf. Appendix). Thus, let us consider the exact sequence of local cohomology in the algebraic category.

$$H^1_{\pi^{-1}(\Sigma_0)}(\Theta^{alg}_{\tilde{X}}) \to H^1(\Theta^{alg}_{\tilde{X}}) \to H^1(\pi^{-1}(U), \Theta^{alg}_{\pi^{-1}(U)}) \to H^2_{\pi^{-1}(\Sigma_0)}(\Theta^{alg}_{\tilde{X}}).$$

We only have to prove that the middle map is an isomorphism by GAGA and by the fact mentioned above. Let X_{Σ_0} (resp. $\tilde{X}_{\Sigma_0}$) be the formal completion of X (resp. $\tilde{X}$) along Σ_0 (resp. $\pi^{-1}(\Sigma_0)$). By duality, we have $H^{n-1}(\tilde{X}_{\Sigma_0}, \Omega^{1,alg}_{\tilde{X}_{\Sigma_0}}) \cong H^1_{\pi^{-1}(\Sigma_0)}(\Theta^{alg}_{\tilde{X}})^*$ and $H^{n-2}(\tilde{X}_{\Sigma_0}, \Omega^{1,alg}_{\tilde{X}_{\Sigma_0}}) \cong H^2_{\pi^{-1}(\Sigma_0)}(\Theta^{alg}_{\tilde{X}})^*$.

Note that $\Omega^{1,alg}_{\tilde{X}} \cong \Omega^{n-1,alg}_{\tilde{X}}$ by a non-degenerate 2-form ω. Therefore, we have to prove that $H^{n-1}(\tilde{X}_{\Sigma_0}, \Omega^{n-1,alg}_{\tilde{X}_{\Sigma_0}}) = H^{n-2}(\tilde{X}_{\Sigma_0}, \Omega^{n-1,alg}_{\tilde{X}_{\Sigma_0}}) = 0$. We shall prove that $R^k\pi_*\Omega^{n-1,alg}_{\tilde{X}} = 0$ if $k \geq 2$. If these are proved, then by the Leray spectral sequence

$$E_2^{p,q} = H^p(X_{\Sigma_0}, R^q\pi_*\Omega^{n-1,alg}_{\tilde{X}}) => H^{p+q}(\tilde{X}_{\Sigma_0}, \Omega^{n-1,alg}_{\tilde{X}_{\Sigma_0}})$$

we have $H^{n-1}(\tilde{X}_{\Sigma_0}, \Omega^{n-1,alg}_{\tilde{X}_{\Sigma_0}}) = H^{n-2}(\tilde{X}_{\Sigma_0}, \Omega^{n-1,alg}_{\tilde{X}_{\Sigma_0}}) = 0$ because $\dim \Sigma_0 \leq n-4$ by Proposition (1.6).

By GAGA, it suffices to show, in the analytic category, that $R^k\pi_*\Omega^{n-1}_{\tilde{X}} = 0$ if $k \geq 2$.

Let $\nu : Y \to \tilde{X}$ be a composition of blowing ups with smooth centers such that the total transform of $\mathrm{Exc}(\pi)$ is a divisor with normal crossings. Set $f := \pi \circ \nu$. We put $E := \mathrm{Exc}(f)$ and $E' := \mathrm{Exc}(\nu)$.

Claim 1. $R^k\pi_*\Omega^{n-1}_{\tilde{X}} \cong R^kf_*\Omega^{n-1}_Y(logE')(-E')$.

Proof. By [St 2], $R^l\nu_*\Omega^{n-1}_Y(logE')(-E') = 0$ for $l \geq 2$. The same statement also holds when $l = 1$. The proof of this fact is the following. Since $\nu_*\hat{\Omega}^{n-1}_{E'} = 0$, we have an exact sequence

$$0 \to R^1\nu_*\Omega^{n-1}_Y(logE')(-E') \to R^1\nu_*\Omega^{n-1}_Y \xrightarrow{\gamma'} R^1\nu_*\hat{\Omega}^{n-1}_{E'}.$$

The map γ' is an isomorphism because $\tilde{X}$ is smooth and ν is a composition of the blowing-ups with smooth centers. In fact, at first, by the exact sequence

$$R^1\nu_*\Omega_Y^{n-1} \to R^1\nu_*\hat{\Omega}_{E'}^{n-1} \to R^2\nu_*\Omega_Y^{n-1}(logE')(-E')$$

γ' is surjective since the last term vanishes by [St 2]. Assume that ν is a composition of exactly k blowing-ups. We shall prove that γ' is an isomorphism by the induction on k. We can check it directly when $k = 1$. Assume that $k > 1$. Decompose ν as $Y \overset{\nu_2}{\to} Y_1 \overset{\nu_1}{\to} \tilde{X}$ in such a way that ν_1 is a blowing-up with a smooth center. Let $E_1 := \mathrm{Exc}(\nu_1)$. Then the proper transform E_1' of E_1 by ν_2 is an irreducible component of $E' := \mathrm{Exc}(\nu)$. Let $E' = \Sigma E'_i$ be the irreducible decomposition. We see that $\nu_{2*}\hat{\Omega}_{E'}^{n-1} \cong \oplus\nu_{2*}\Omega_{E'_i}^{n-1} \cong \nu_{2*}\Omega_{E_1'}^{n-1} \cong \Omega_{E_1}^{n-1}$. There is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & R^1\nu_{1*}\Omega_{Y_1}^{n-1} & \longrightarrow & R^1\nu_*\Omega_Y^{n-1} & \longrightarrow & \nu_{1*}R^1\nu_{2*}\Omega_Y^{n-1} \\ & & \downarrow & & \gamma'\downarrow & & \downarrow \\ 0 & \longrightarrow & R^1\nu_{1*}\hat{\Omega}_{E_1}^{n-1} & \longrightarrow & R^1\nu_*\hat{\Omega}_{E'}^{n-1} & \longrightarrow & \nu_{1*}R^1\nu_{2*}\hat{\Omega}_{E'}^{n-1} \end{array} \tag{4}$$

The vertical maps except γ' are both isomorphisms by the induction, hence γ' is injective by the diagram. Thus γ' is an isomorphism.

Now by a Leray spectral sequence, $R^kf_*\Omega_Y^{n-1}(logE')(-E') \cong R^k\pi_*(\nu_*\Omega_Y^{n-1}(logE')(-E'))$. By the exact sequence

$$0 \to \nu_*\Omega_Y^{n-1}(logE')(-E') \to \nu_*\Omega_Y^{n-1} \to \nu_*\hat{\Omega}_{E'}^{n-1} = 0$$

we see that $\nu_*\Omega_Y^{n-1}(logE')(-E') \cong \nu_*\Omega_Y^{n-1}$, where the second term is isomorphic to $\Omega_{\tilde{X}}^{n-1}$ (cf. [St 1]). Q.E.D.

Claim 2. *$R^kf_*\Omega_Y^{n-1}(logE')(-E') = 0$ if $k \geq 2$.*

Proof. Consider the exact sequence

$$R^{k-1}f_*\Omega_Y^{n-1} \overset{\alpha}{\to} R^{k-1}f_*\hat{\Omega}_{E'}^{n-1} \to R^kf_*\Omega_Y^{n-1}(logE')(-E') \to R^kf_*\Omega_Y^{n-1} \overset{\beta}{\to} R^kf_*\hat{\Omega}_{E'}^{n-1}.$$

The map α is factorized as $R^{k-1}f_*\Omega_Y^{n-1} \to R^{k-1}f_*\hat{\Omega}_E^{n-1} \to R^{k-1}f_*\hat{\Omega}_{E'}^{n-1}$, and the second map is clearly a surjection. The first map is also surjective by the exact sequence

$$R^{k-1}f_*\Omega_Y^{n-1} \to R^{k-1}f_*\hat{\Omega}_E^{n-1} \to R^kf_*\Omega_Y^{n-1}(logE)(-E)$$

because $R^kf_*\Omega_Y^{n-1}(logE)(-E) = 0$ for $k \geq 2$ by [St 2]. Hence α is a surjection.

The map β is similarly factorized as $R^kf_*\Omega_Y^{n-1} \to R^kf_*\hat{\Omega}_E^{n-1} \to R^kf_*\hat{\Omega}_{E'}^{n-1}$. Note that the second map is an isomorphism. Indeed, let E'' be an f-exceptional

divisor which is not contained in E'. Then, by Corollary (1.5), E'' is mapped to an (n-2)-dimensional subvariety of X by f; in particular, a general fiber of $E'' \to f(E'')$ has dimension 1. By [Ko 2] $R^k f_* \hat{\Omega}^{n-1}_{E''} = 0$ if $k \geq 2$. Hence $R^k f_* \hat{\Omega}^{n-1}_E \cong R^k f_* \hat{\Omega}^{n-1}_{E'}$.

The first map is injective by the exact sequence

$$R^k f_* \Omega^{n-1}_Y (log E)(-E) \to R^k f_* \Omega^{n-1}_Y \to R^k f_* \hat{\Omega}^{n-1}_E$$

because the first term vanishes by [St 2]. Hence β is an injection. Q.E.D.

Theorem (2.2). *Let $\pi : \tilde{X} \to X$ be a symplectic resolution of a projective symplectic variety X of dimension n. Then the Kuranishi spaces* $\mathrm{Def}(\tilde{X})$ *and* $\mathrm{Def}(X)$ *are both smooth of the same dimension. The natural map* $\pi_* : \mathrm{Def}(\tilde{X}) \to \mathrm{Def}(X)$ *is a finite covering. Moreover, X has a flat deformation to a symplectic n-fold X_t. Any smoothing X_t of X is obtained as a flat deformation of $\tilde{X}$.*

Proof. By Bogomolov [Bo] the Kuranishi space $\mathrm{Def}(\tilde{X})$ is smooth for a symplectic manifold $\tilde{X}$.

Claim 1. $\dim \mathbf{T}^1_X \leq \dim \mathrm{Def}(\tilde{X})$.

Proof. Since $\mathrm{Def}(\tilde{X})$ is smooth, we have to prove that $\dim \mathbf{T}^1_X \leq h^1(\tilde{X}, \Theta_{\tilde{X}})$. By Proposition (2.1) it suffices to prove that $\dim \mathbf{T}^1_U \leq h^1(\pi^{-1}(U), \Theta_{\pi^{-1}(U)})$. We shall use the same notation as (1.8). Recall that $\pi_U := \pi|_{\tilde{U}}$, $D_U := \mathrm{Exc}(\pi_U)$ and $D_U = D_1 \cup ... \cup D_m$ is the irreducible decomposition.

(i) By Corollary (1.10) we have $h^0(U, T^1_U) \leq m$.

(ii) Consider the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(U, \Theta_U) & \longrightarrow & H^1(\tilde{U}, \Theta_{\tilde{U}}) & \xrightarrow{\tilde{\eta}} & H^0(U, R^1(\pi_U)_* \Theta_{\tilde{U}}) \\ & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(U, \Theta_U) & \longrightarrow & \mathbf{T}^1_U & \xrightarrow{\eta} & H^0(U, T^1_U) \end{array} \qquad (5)$$

We shall prove that $\tilde{\eta}$ is surjective and prove that $h^0(U, R^1(\pi_U)_* \Theta_{\tilde{U}}) = m$, where m is the number of irreducible components of D_U.

By a non-degenerate 2-form ω on $\tilde{X}$, we have $H^0(U, R^1(\pi_U)_* \Theta_{\tilde{U}}) \cong H^0(U, R^1(\pi_U)_* \Omega^1_{\tilde{U}})$.

By the definition of U we know that $R^1(\pi_U)_* \Omega^1_{\tilde{U}} \cong R^1(\pi_U)_* \hat{\Omega}^1_{D_U}$. There is an exact sequence

$$\oplus_i (\pi_U)_* \Omega^1_{\tilde{D}_i} \to \oplus_{i>j} (\pi_U)_* \Omega^1_{\tilde{D}_{i,j}} \to R^1(\pi_U)_* \hat{\Omega}^1_{D_U} \to \oplus_i R^1(\pi_U)_* \Omega^1_{\tilde{D}_i}$$

where $\tilde{D}_i$ and $\tilde{D}_{i,j}$ are normalizations of D_i and $D_{i,j}$ respectively.

The first map is surjective by the description of D_U in (i). The last map is a surjection (and hence an isomorphism) because $R^1(\pi_U)_* \Omega^1_{\tilde{D}_{i,j}} = 0$. Let

$\tilde{D}_i \xrightarrow{\pi_i} S_i \xrightarrow{\varphi_i} \Sigma \setminus \Sigma_0$ be the Stein factorization. Since φ_i is finite, $R^1(\pi_U)_*\Omega^1_{\tilde{D}_i} \cong (\varphi_i)_* R^1(\pi_i)_*\Omega^1_{\tilde{D}_i}$. Since $\tilde{D}_i$ is a $\mathbf{P}^1$-bundle over S_i, we have $R^1(\pi_i)_*\Omega^1_{\tilde{D}_i} \cong \mathcal{O}_{S_i}$. Thus $H^0(U, R^1(\pi_U)_*\Omega^1_{\tilde{D}_i}) \cong H^0(U, \phi_{i*}\mathcal{O}_{S_i}) = H^0(S_i, \mathcal{O}_{S_i}) = \mathbf{C}$.

As a consequence, $h^0(U, R^1(\pi_U)_*\Theta_{\tilde{U}}) = m$.

(iii) By a non-degenerate 2-form ω on $\tilde{X}$, the map $\tilde{\eta}$ is identified with the map $H^1(\tilde{U}, \Omega^1_{\tilde{U}}) \to H^0(U, R^1(\pi_U)_*\Omega^1_{\tilde{U}})$. We have a commutative diagram

$$\begin{array}{ccc} H^1(\tilde{U}, \mathcal{O}^*_{\tilde{U}}) & \longrightarrow & H^0(U, R^1(\pi_U)_*\mathcal{O}^*_{\tilde{U}}) \\ {\scriptstyle dlog}\downarrow & & {\scriptstyle dlog}\downarrow \\ H^1(\tilde{U}, \Omega^1_{\tilde{U}}) & \longrightarrow & H^0(U, R^1(\pi_U)_*\Omega^1_{\tilde{U}}) \end{array} \tag{6}$$

The $\mathbf{C}$-vector space $H^0(U, R^1(\pi_U)_*\Omega^1_{\tilde{U}})$ is generated by the images of $[D_i] \in H^1(\tilde{U}, \mathcal{O}^*_{\tilde{U}})$ $(1 \leq i \leq m)$ by (ii). Thus the horizontal map at the bottom is surjective, and $\tilde{\eta}$ is also surjective.

(iv) By the commutative diagram (6) $h^0(\tilde{U}, \Theta_{\tilde{U}}) = h^0(U, \Theta_U) + m$ because $h^0(U, R^1(\pi_U)_*\Theta_{\tilde{U}}) = m$ and $\tilde{\eta}$ is surjective. On the other hand, $dim \mathbf{T}^1_U \leq h^0(U, \Theta_U) + m$ because $h^0(U, T^1_U) \leq m$. These imply that $\dim \mathbf{T}^1_U \leq h^1(\tilde{U}, \Theta_{\tilde{U}})$.

Claim 2. *The map $\pi_* : \mathrm{Def}(\tilde{X}) \to \mathrm{Def}(X)$ has finite fibers.*

Proof. Let $\Pi : \tilde{\mathcal{X}} \to X \times \Delta^1$ be a flat deformation of the map $\pi : \tilde{X} \to X$ over a 1-dim disc Δ^1. We have to show that $\tilde{\mathcal{X}} \cong \tilde{X} \times \Delta^1$. Let $S_n := \mathrm{Spec}\mathbf{C}[t]/(t^{n+1})$ and let $\tilde{X}_n$ be the pull back of $\tilde{\mathcal{X}}$ by the natural embedding $S_n \to \Delta^1$. We have to prove that $\tilde{X}_n \cong \tilde{X} \times S_n$ for all n. By Proposition (2.1), there is a one to one corrspondence between infinitesimal deformations of $\tilde{X}$ and infinitesimal deformations of $\tilde{U} := \pi^{-1}(U)$. Therefore, it suffices to prove the same statement by replacing X by U and $\tilde{X}$ by $\tilde{U}$ respectively. But, then $\tilde{U}_n$ should be a (relatively) minimal resolution of $U \times S_n$. By the uniqueness of minimal resolution, we have $\tilde{U}_n \cong \tilde{U} \times S_n$. Q.E.D.

By Claims 1 and 2, $\dim \mathrm{Def}(\tilde{X}) = \dim \mathbf{T}^1_X$. Since $\mathrm{Def}(\tilde{X})$ is smooth by [Bo], $\mathrm{Def}(X)$ is also smooth. Moreover, π_* is a finite covering (cf. [Fi, 3.2, p 132]).

Claim 3. *X has a flat deformation to a smooth symplectic n-fold X_t such that X_t is a small deformation of $\tilde{X}$.*

Proof. The proof is due to [Fu, (3) in the proof of Theorem (5.7)]. By the existence of a non-degenerate 2-form ω, there is an obstruction to extending a holomorphic curve on $\tilde{X}$ sideways in a given one-parameter small deformation $\tilde{\mathcal{X}} \to \Delta^1$. Therefore, if we take a general curve of $\mathrm{Def}(\tilde{X})$ passing through the origin and take a corresponding small deformation of $\tilde{X}$, then no holomorphic curves survive. A detailed argument on this fact can be found in [Fu, Theorem

(4.8), (1)]; the theorem assumes that $\tilde{X}$ is primitively symplectic, however, one can prove the same result in a general case by a minor modification.

Let $t \in \mathrm{Def}(X)$ be a generic point (that is, t is outside the union of a countable number of proper subvarieties of $\mathrm{Def}(X)$). Since $\pi_* : \mathrm{Def}(\tilde{X}) \to \mathrm{Def}(X)$ is a finite covering, we may assume that X_t has a symplectic resolution $\pi_t : \tilde{X}_t \to X_t$. By the argument above, $\tilde{X}_t$ contains no curves. By Chow lemma [Hi], there is a bimeromorphic projective map $h : W \to X_t$ such that h is factored through π_t. Since $h^{-1}(p)$ is the union of projective varieties for $p \in X_t$, ${\pi_t}^{-1}(p)$ is the union of Moishezon varieties. If π_t is not an isomorphism, then ${\pi_t}^{-1}(p)$ has positive dimension for some point $p \in X_t$; hence $\tilde{X}_t$ contains curves, which is a contradiction. Thus π_t is an isomorphism and X_t is a (smooth) symplectic n-fold.

Example (2.4). As compared with Calabi-Yau 3-folds, the statement of (2.2) for symplectic varieties is quite simple. We shall briefly discuss the difference by comparing some examples.

Let us consider the situation where $\tilde{X}$ is a smooth Calabi-Yau 3-fold containing a smooth divisor E and $\pi : \tilde{X} \to X$ is a projective birational contraction map of E to a curve $C \subset X$. We assume that C is a smooth curve and $E \to C$ is a conic bundle with no multiple fibers. Denote by g the genus of C and denote by n the number of the singular fibers of the conic bundle. Such birational contractions are studied in [Wil, Gr, Na 4].

X has A_1 singularities along C; hence $\Sigma = \mathrm{Sing}(X)$ is isomorphic to C. Corresponding to n singular fibers, X has exactly n dissident points.

First note that any g is possible; hence ω_C is not necessarily trivial (Compare with the symplectic case (1.6)). Moreover, the dissident locus Σ_0 has codimension 3. Proposition (2.1) is no more true; $\mathbf{T}^1_X$ and $\mathbf{T}^1_U$ are isomorphic, but $H^1(\pi^{-1}(U), \Theta_{\pi^{-1}(U)})$ is an infinitesimal dimensional $\mathbf{C}$ vector space.

Assume that $g = 0$ (i.e. $C = \mathbf{P}^1$) and $n \geq 3$. Then $\mathrm{Def}(\tilde{X})$ and $\mathrm{Def}(X)$ are both smooth, but $\dim \mathrm{Def}(X) = \dim \mathrm{Def}(\tilde{X}) + 2n - 2 - k$, where $k := b_2(\tilde{X}) - b_2(X)$. By the natural map π_*, $\mathrm{Def}(\tilde{X})$ is embedded into $\mathrm{Def}(X)$. However, $\dim \mathrm{Def}(X) > \dim \mathrm{Def}(\tilde{X})$ if $n \geq 4$.

When $g = 1$ and $n = 0$, we are in a similar situation to (2.2), that is, $\mathrm{Def}(\tilde{X})$ and $\mathrm{Def}(X)$ are both smooth and π_* is a finite covering.

In Theorem (2.2) we have studied a singular variety X which has a symplectic resolution. However we must often deal with a *symplectic* variety which does not have a symplectic resolution; for example, such varieties appeared in [O] as the moduli spaces of rank 2 semi-stable sheaves on a K3 surface with $c_1 = 0$ and with even $c_2 \geq 6$. (When $c_1 = 0$ and $c_2 = 4$, the moduli space has a 10 dimensional symplectic resolution and it provides us with a new example of a symplectic manifold.) Finally we shall prove an unobstructedness result for such singular symplectic varieties.

Theorem (2.5). *Let X be a projective symplectic variety. Let $\Sigma \subset X$ be*

the singular locus. Assume that $\mathrm{codim}(\Sigma \subset \mathrm{X}) \geq 4$. *Then* $\mathrm{Def}(X)$ *is smooth.*

Remark. When X has a symplectic resolution $\pi : \tilde{X} \to X$, the result easily follows from Proposition (2.1) because $\mathrm{Def}(\tilde{X})$ is smooth by a theorem of Bogomolov.

Proof. The following result of Ohsawa [Oh] is a key. We shall give an algebraic proof.

Lemma (2.6). *The Hodge spectral sequence*

$$E_1^{p,q} = H^q(U, \Omega_U^p) => H^{p+q}(U, \mathbf{C})$$

degenerates at E_1*-terms with* $p+q=2$.

Proof of Lemma (2.6). Let $f : Y \to X$ be a resolution of singularities such that $E := f^{-1}(\Sigma)$ is a divisor with normal crossings and $f^{-1}(U) \cong U$. There are natural maps $\phi_{p,q} : H^q(Y, \Omega_Y^p(\log E)) \to H^q(U, \Omega_U^p)$ induced by the restriction. By the mixed Hodge structure on $H^*(U, \mathbf{C})$, the spectral sequence

$$E^1_{p,q} := H^q(Y, \Omega_Y^p(\log E)) => H^{p+q}(Y, \Omega_Y^{\cdot}(\log E)) = H^{p+q}(U, \mathbf{C})$$

degenerates at E_1 terms. Therefore we only have to show that $\phi_{p,q}$ are isomorphisms for p, q with $p+q=2$.

By Appendix we will see that $H^q(U, \Omega_U^{p,alg}) \cong H^q(U, \Omega_U^p)$ when $p+q=2$. Thus, we have to prove that $\phi_{p,q}^{alg} : H^q(Y, \Omega_Y^{p,alg}(\log E)) \to H^q(U, \Omega_U^{p,alg})$ are isomorphisms when $p+q=2$. By the local cohomology sequences, it is enough to show that

(1) $H^i_E(Y, \mathcal{O}_Y^{alg}) = 0$ for $i = 2, 3$,

(2) $H^i_E(Y, \Omega_Y^{1,alg}(\log E)) = 0$ for $i = 1, 2$, and

(3) $H^i_E(Y, \Omega_Y^{2,alg}(\log E)) = 0$ for $i = 0, 1$.

For (1) let us show that $H^3_E(Y, \mathcal{O}_Y^{alg}) = 0$. Let Y_Σ be the formal completion of Y along $f^{-1}(\Sigma)$ and X_Σ the formal completion of X along Σ. $H^3_E(Y, \mathcal{O}_Y^{alg})$ is dual to $H^{n-3}(Y_\Sigma, \omega_{Y_\Sigma}^{alg})$. Note that $H^{n-3}(X_\Sigma, f_*\omega_{Y_\Sigma}^{alg}) = 0$ because $\mathrm{codim}(\Sigma \subset \mathrm{X}) \geq 4$. By the Grauert-Riemenschneider vanishing theorem and GAGA principle, $R^i f_*\omega_{Y_\Sigma}^{alg} = 0$ for $i > 0$. Therefore $H^{n-3}(Y_\Sigma, \omega_{Y_\Sigma}^{alg}) = 0$. The proof that $H^2_E(Y, \mathcal{O}_Y^{alg}) = 0$ is similar.

Next we shall prove (2).

$H^2_E(Y, \Omega_Y^{1,alg}(\log E))$ is dual to $H^{n-2}(Y_\Sigma, \Omega_Y^{n-1,alg}(\log E)(-E))$.

Since $\mathrm{Codim}(\Sigma \subset \mathrm{X}) \geq 4$, the vector spaces $H^{n-2}(X_\Sigma, f_*\Omega_Y^{n-1,alg}(\log E)(-E))$ and $H^{n-3}(X_\Sigma, R^1 f_*\Omega_Y^{n-1,alg}(\log E)(-E)) = 0$ are both zero.

On the other hand, $R^i f_*\Omega_Y^{n-1,alg}(\log E)(-E) = 0$ for $i \geq 2$ by [St 2] and GAGA principle. Therefore $H^{n-2}(Y_\Sigma, \Omega_Y^{n-1,alg}(\log E)(-E)) = 0$. We can show that $H^1_E(Y, \Omega_Y^{1,alg}(\log E)) = 0$ in a similar way.

The proof of (3) is similar to that of (2).

Lemma (2.7). *Let X_m be an infinitesimal deformation of X over $S_m :=$ $\mathrm{Spec(A_m)}$, where $A_m = \mathbf{C}[t]/(t^{m+1})$. Let $U_m := X_m|_U$. Then the Hodge spectral sequence*

$$E_1^{p,q} = H^q(U, \Omega^p_{U_m/S_m}) => H^{p+q}(U, A_m)$$

degenerates at E_1-terms with $p+q=2$. In particular, the natural map $H^q(U, \Omega^p_{U_m/S_m}) \to H^q(U, \Omega^p_{U_{m-1}/S_{m-1}})$ is surjective, where $U_{m-1} := U_m \times_{S_m} S_{m-1}$.

Proof. Note that $H^{p+q}(U, A_m) \cong H^{p+q}(U, \mathbf{C}) \otimes_{\mathbf{C}} A_m$. When $m=0$ the Hodge spectral sequence degenerates at E_1-terms with $p+q=2$ by Lemma (2.6). Hence $\Sigma_{p+q=2} \dim_{\mathbf{C}} H^q(U, \Omega^p_{U_m/S_m}) = \dim_{\mathbf{C}} H^2(U, A_m)$. From this it follows that $H^q(U, \Omega^p_{U_m/S_m})$ are free A_m modules for p, q with $p+q=2$ and the Hodge sectral sequence degenerates at E_1-terms with $p+q=2$. Q.E.D.

By Lemma (2.7) the non-degenerate holomorphic 2-form ω extends to an relative 2-form $\omega_m \in H^0(U, \Omega^2_{U_m/S_m})$, by which Θ_{U_m/S_m} and $\Omega^1_{U_m/S_m}$ are identified. Now again by Lemma (2.7) the natural map $H^1(U, \Theta_{U_m/S_m}) \to H^1(U, \Theta_{U_{m-1}/S_{m-1}})$ is surjective. By Proposition (2.1) this implies that the T^1-lifting property holds for an infinitesimal deformation of X. Therefore $\mathrm{Def}(X)$ is smooth. Q.E.D.

3. Appendix.

We shall prove two comparison theorems.

Lemma (3.1). (1) *Let $\pi : \tilde{X} \to X$ be a birational projective morphism from a smooth symplectic n-fold to a normal projective variety X. Let U be the same as (1.8) and put $\tilde{U} := \pi^{-1}(U)$. Then*

$$H^1(\tilde{U}, \Theta^{alg}_{\tilde{U}}) \cong H^1(\tilde{U}, \Theta_{\tilde{U}}).$$

(2) *Let X be a projective variety with* $\mathrm{Codim}(\Sigma \subset \mathrm{X}) \geq 4$, *where Σ is the singular locus of X. Let U be the regular part of X. Then*

$$H^q(U, \Omega^{p,alg}_U) \cong H^q(U, \Omega^p_U)$$

for p and q with $p+q=2$.

Proof. (1): We shall use the same notation as (1.8). There are two exact sequences in the algebraic/ anlytic category:

$$0 \to H^1(\Theta^{alg}_U) \to H^1(\Theta^{alg}_{\tilde{U}}) \to H^0(U, R^1(\pi_U)_*\Theta^{alg}_{\tilde{U}}) \to H^2(\Theta^{alg}_{\tilde{U}})$$

and

$$0 \to H^1(U, \Theta_U) \to H^1(\Theta_{\tilde{U}}) \to H^0(U, R^1(\pi_U)_*\Theta_{\tilde{U}}) \to H^2(\Theta_{\tilde{U}}).$$

There are natural maps from the first sequence to the second one so that they make a commutative diagram. Call these maps α_1, α_2, α_3 and α_4 from the left to the right. We want to prove that α_2 is an isomorphism.

By a symplectic 2-form ω, one has an isomorphism $\Theta_{\tilde{U}}^{alg} \cong \Omega_{\tilde{U}}^{1,alg}$ (resp. $\Theta_{\tilde{U}} \cong \Omega^1_{\tilde{U}}$).

By taking the direct image of both sides, one has $\hat{\Omega}^1_U \cong \Theta_U$, where ˆ means the double dual (cf. [St 1]). Since U has only quotient singularities, $\mathrm{depth}(\hat{\Omega}^1_{\mathrm{U}})_{\mathrm{p}} = \mathrm{n}$ for $p \in U$. Therefore, $\mathrm{depth}(\Theta_{\mathrm{U}})_{\mathrm{p}} = \mathrm{n}$. Now, since $\mathrm{Codim}(\Sigma_0 \subset \mathrm{X}) \geq 4$ where $\Sigma_0 = X - U$, the maps α_1 and α_4 are isomorphisms. (cf. [Ha, VI, Theorem 2.1, (a)])[4]

We shall prove that α_3 is an isomorphism.

At (i) of the proof of Claim 1 in Theorem (2.2), we have shown that $R^1\pi_U * \Omega^1_{\tilde{U}} \cong R^1\pi_U\Omega^1_{D_U}$. In particular, we see that $R^1\pi_{U*}\Omega^1_{\tilde{U}}$ is a locally free $\mathcal{O}_{\Sigma^{(0)}}$ module, hence $R^1\pi_{U*}\Theta_{\tilde{U}}$ is also a locally free $\mathcal{O}_{\Sigma^{(0)}}$ module. $\Sigma^{(0)}$ is a Zariski open set of Σ and the complement $\Sigma - \Sigma^{(0)}$ has codmension at least 2 by (1.6).

Since $R^1\pi_*\Theta^{alg}_{\tilde{X}} \otimes_{\mathcal{O}^{alg}_X} \mathcal{O}^{alg}_\Sigma|_U = R^1\pi_{U*}\Theta^{alg}_{\tilde{U}}$ and $R^1\pi_*\Theta_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_\Sigma|_U = R^1\pi_{U*}\Theta_{\tilde{U}}$, we conclude that α_3 is an isomorphism by [Ha, VI, Theorem 2.1].

Now, by the commutative diagram above, α_2 is an isomorphism, which completes the proof of (1).

(2): This easily follows from [Ha, VI, Theorem 2.1, (a)].

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